Lemma Let $\mathcal{Y} = \mathrm{codisc}\{0,1\} = \begin{tikzcd} 0 \arrow[loop above] \arrow[r, bend left] & 1 \arrow[loop above] \arrow[l, bend left] \end{tikzcd}$. Consider the pushout

$$\begin{tikzcd} \{0,1\} \arrow[r, hook] \arrow[d, hook] & \mathcal{Y} = 0 \underset{a^{-1}}{\overset{a}{\rightleftarrows}} 1 \arrow[d, hook] \\ 0 \underset{b^{-1}}{\overset{b}{\rightleftarrows}} 1 = \mathcal{Y} \arrow[r] & \Delta \end{tikzcd}$$

Then $\Delta \simeq B\mathbb{Z}$.

Proof. Recall: $\mathrm{Hom}_\Delta(0,0) = \left\{ \begin{matrix} 0 \xrightarrow{f_1} 1 \xrightarrow{f_2} 0 \to \cdots \xrightarrow{f_n} 0 \\ n \geq 0, \quad f_i \in \{a, a^{-1}, b, b^{-1}\} \end{matrix} \right\} \Big/$ we can cancel aa^{-1}, $a^{-1}a$, bb^{-1}, $b^{-1}b$

Any such sequence is equivalent to a concatenation of

$$0 \xrightarrow{a} 1 \xrightarrow{b^{-1}} 0$$
$$0 \xrightarrow{b} 1 \xrightarrow{a^{-1}} 0$$

These are inverse to one another, so $0 \xrightarrow{a} 1 \xrightarrow{b^{-1}} 0$ generates $\mathrm{Hom}_\Delta(0,0)$ as a group.

$\Rightarrow$ We have a surjective group homomorphism $\mathbb{Z} \twoheadrightarrow \mathrm{Hom}_\Delta(0,0)$
$1 \longmapsto (0 \xrightarrow{a} 1 \xrightarrow{b^{-1}} 0)$

Define $\mathrm{Hom}_\Delta(0,0) \longrightarrow \mathbb{Z}$

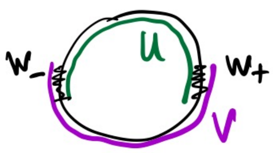
$$[f_1, f_2, \dots, f_n] \longmapsto (\#\text{ of } a) - (\#\text{ of } a^{-1}).$$

Note: • well-defined
• group homomorphism

• $\mathbb{Z} \longrightarrow \mathrm{Hom}_\Delta(0,0) \longrightarrow \mathbb{Z}$.
$1 \longmapsto (0 \xrightarrow{a} 1 \xrightarrow{b^{-1}} 0) \longmapsto 1$
$\Rightarrow \mathbb{Z} \xrightarrow{\cong} \mathrm{Hom}_\Delta(0,0)$. $\square$

Theorem $\pi_1(S^1, 1) \cong \mathbb{Z}$ with generator $[s \mapsto e^{2\pi i s}]_p$.

Proof



$U \cap V = W_+ \amalg W_-$.

By the SvK theorem, we have a pushout

$$\begin{tikzcd} \pi_1(W_+) \amalg \pi_1(W_-) \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\mathrm{PO}"] & \pi_1(U) \arrow[d] \\ \pi_1(V) \arrow[r] & \pi_1(S^1) \end{tikzcd}$$

U, V, W_+, W_- are contractible, so $\Pi_1 \simeq *$.

Let $\mathcal{Y}$ = codisc $\{1,-1\}$. We have a commutative diagram:

$$\begin{tikzcd}
\{\pm 1\} \arrow[r] \arrow[d] \arrow[dr, hook, "\simeq"] & \mathcal{Y} \arrow[dr, hook, "\simeq"] & & \\
\mathcal{Y} \arrow[dr, hook, "\simeq"] & \Pi_1(W_+) \amalg \Pi_1(W_-) \arrow[r] \arrow[d] \arrow[dr, "!"] & \Pi_1(U) \arrow[dr] & \\
& \Pi_1(V) \arrow[dr] & \{\pm 1\} \arrow[r] \arrow[d] & \mathcal{Y} \\
& & \mathcal{Y} &
\end{tikzcd}$$

$x \longmapsto \begin{cases} 1 & \text{if } Re(x) > 0 \\ -1 & \text{if } Re(x) \leq 0 \end{cases}$ (for $\Pi_1(U) \to \mathcal{Y}$)

$x \longmapsto \begin{cases} 1 & \text{if } Re(x) > 0 \\ -1 & \text{if } Re(x) \leq 0 \end{cases}$ (for $\Pi_1(V) \to \mathcal{Y}$)

The diagonal compositions are the identity. Taking pushouts, we get

$$\begin{tikzcd}
\Delta \arrow[r] \arrow[rr, bend right, "id"'] & \Pi_1(S^1) \arrow[r] & \Delta
\end{tikzcd}$$

By the lemma $\Delta \simeq B\mathbb{Z}$. This shows that the group homomorphism

$$\mathbb{Z} \longrightarrow \mathrm{Hom}_{\Pi_1(S^1)}(1,1), \quad 1 \longmapsto [-1 \frown 1 * {}^{-1} \smile {}^{1}]_p = [s \mapsto e^{2\pi i s}]_p.$$

is injective.

Surjectivity: $\mathrm{Hom}_{\Pi_1(S^1)}(1,1) = \left\{ 1 = x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} x_2 \to \cdots \xrightarrow{f_n} x_n = 1 \;\middle|\; f_i \in \mathrm{Mor}\, \Pi_1(U) \amalg \mathrm{Mor}\, \Pi_1(V) \right\} \big/ \text{relations}$

Using the relations, we can assume that $x_i \in \{\pm 1\}$.

Then such a sequence is in the image of the functor

$$\Delta \to \Pi_1(S^1). \qquad \square$$

<u>Corollary</u>. Let $T^n = (S^1)^n$ be the n-dimensional torus.

$\pi_1(T^n, \cdot) \cong \mathbb{Z}^n$.

<u>Pf</u> π_1 preserves products.

Theorem If $n \geq 2$, S^n is simply path-connected, i.e., $\pi_1(S^n, \cdot) = 0$.

Pf. Write $S^n = U \cup V$, $U = S^n -$ north pole
$V = S^n -$ south pole

$U \cap V \cong S^{n-1} \times (0,1)$. is path-connected since $n \geq 2$

By the SvK theorem, we have a pushout of groups

$$\begin{array}{ccc} \pi_1(S^{n-1}) & \longrightarrow & \pi_1(U) = 0 \\ \downarrow & & \downarrow \\ 0 = \pi_1(V) & \longrightarrow & \pi_1(S^n) \end{array} \implies \pi_1(S^n) = 0 \qquad \square$$

Applications.

Fundamental theorem of algebra

Let $p(z) \in \mathbb{C}[z]$ be a non constant polynomial.
Then $p(z)$ has a zero in $\mathbb{C}$.

Proof. Write $p(z) = z^n + q(z)$ where $n = \deg(p)$.

Let $h(z,t) = z^n + tq(z)$.

Choose $r \gg 0$ such that $|z^n| > |q(z)|$ for $|z| \geq r$.

Then $h(z,t) \neq 0$ for $|z| \geq r$ and $t \in [0,1]$

Define
$$H: S^1 \times I \longrightarrow \mathbb{C} \setminus \{0\}$$
$$(z,t) \longmapsto \frac{h(rz,t)}{h(r,t)}$$

Then H satisfies:
$$H(z,0) = z^n$$
$$H(z,1) = \frac{p(rz)}{p(r)}$$
$$H(1,t) = 1$$

$\Rightarrow$ H is a path-homotopy between loops at 1 in $\mathbb{C} \setminus \{0\}$.

$\Rightarrow$ $[H(-,0)]_p = [H(-,1)]_p$ in $\pi_1(\mathbb{C} \setminus \{0\}, 1)$

Under the isomorphism $\pi_1(\mathbb{C} \setminus \{0\}, 1) \cong \mathbb{Z}$,
$$[H(-,0)]_p \leftrightarrow n$$

If $p(z)$ has no zero, so $p: \mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$, then $H(-,1): S^1 \longrightarrow \mathbb{C} \setminus \{0\}$ extends to $\mathbb{C} \simeq *$, so $[H(-,1)]_p = [c_1]_p$. $\Rightarrow$ $0 = n$ in $\mathbb{Z}$.

Contradiction. $\square$

<u>Brouwer fixed point theorem</u> for D^2.

Every continuous map $f: D^2 \longrightarrow D^2$ has a fixed point.

<u>Proof</u>. Exercise.

§4. <u>Covering spaces</u>

<u>Definition</u> Let $\mathcal{F}$ be a collection of topological spaces. A continuous map $p: E \longrightarrow B$ is called a <u>locally trivial bundle</u> (or <u>fiber bundle</u>) with fibers in $\mathcal{F}$ if:

for every $b \in B$, there exists an open neighborhood U of b and an isomorphism over U

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & F \times U \arrow[dl, "pr_2"] \\ & U & \end{tikzcd}$$

for some $F \in \mathcal{F}$.

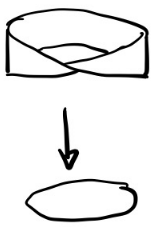
- If this holds for $U = B$, the bundle is called <u>trivial</u>.
- B is called the <u>base</u>, E is called the <u>total space</u> of the bundle
- If $\mathcal{F} = \{F\}$, we say "with fiber F".

<u>Definition</u> A map $p: E \longrightarrow B$ is a <u>covering</u> (or a <u>covering map</u>, or E is a <u>covering space</u> over B) if it is a locally trivial bundle with fibers in Set.

discrete spaces

<u>Notation</u>: Cov_B is the full subcategory of Top/B whose objects are the covering maps.

<u>Examples</u>

- Möbius band: $\begin{matrix} M \\ \downarrow \\ S^1 \end{matrix}$  is a locally trivial bundle with fiber $[0,1]$.

  It is not trivial: $M \not\cong S^1 \times [0,1]$.

- $\mathbb{R}^{n+1}\setminus\{0\} \longrightarrow \mathbb{RP}^n$ is a locally trivial bundle with fiber $\mathbb{R}\setminus\{0\}$

  $\cup$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cup$

  $S^n \longrightarrow \mathbb{RP}^n$ ____ S^0

- $\mathbb{C}^{n+1}\setminus\{0\} \longrightarrow \mathbb{CP}^n$ ____ $\mathbb{C}\setminus\{0\}$

  $\cup$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\cup$

  $S^{2n+1} \longrightarrow \mathbb{CP}^n$ ____ S^1

- If M is a smooth manifold, the tangent bundle $TM \longrightarrow M$ is a locally trivial bundle with fiber $\mathbb{R}^n$ (where $n = \dim M$)

- If G is a Lie group (e.g. $G = GL_n(\mathbb{R})$), if $H \leq G$ is a closed subgroup, then $G \longrightarrow G/H$ is a locally trivial bundle with fiber H.

  For example $GL_{n+1}(\mathbb{R}) \longrightarrow \mathbb{RP}^n$, ...

exercises

- $S^1 \longrightarrow S^1$, $z \mapsto z^n$ $(n \neq 0)$, is a covering map with fiber $\mathbb{Z}/n$.
- $\mathbb{R} \longrightarrow S^1$, $r \mapsto e^{2\pi i r}$ ________________________ $\mathbb{Z}$

  $\mathbb{C} \longrightarrow \mathbb{C}\setminus\{0\}$, $z \mapsto e^z$ ____________________________

- see Hatcher for lots of picture.