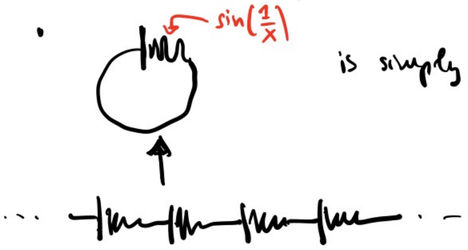
Remarks 1) If B is connected, and $p: E \to B$ is a locally trivial bundle, then all the fibers of p are homeomorphic.

2) The fibers of a covering map can be empty, so a covering is not necessarily surjective.

Definition A top. space X is <u>simply connected</u> if it is nonempty and every covering $E \to X$ is trivial.

Remark simply connected $\Rightarrow$ connected.

Examples:
- S^1 is not simply connected ($\mathbb{R} \not\cong S^1 \times \mathbb{Z}$)
- 

is simply path-connected but not simply connected.

Properties

1) If $p_1: E_1 \to B_1$ and $p_2: E_2 \to B_2$ are coverings then $p_1 \times p_2: E_1 \times E_2 \to B_1 \times B_2$ is a covering.

2) If
$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r] & B \end{tikzcd}$$
is a pullback square in Top where p is a covering, then p' is a covering.

3) If
$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$
is a commutative triangle where p and p' are coverings and B is locally connected, then f is a covering.

Proof: exercise.

Example: $\mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is a covering of the torus, with fiber $\mathbb{Z} \times \mathbb{Z}$

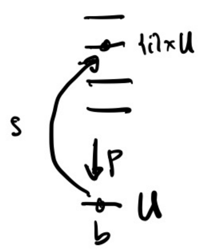
Proposition (Lifting criterion for coverings)

Let
$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dashed, "\tilde{f}"] & B \end{tikzcd}$$
where p is a covering, $e \in E$, $x \in X$ such that $p(e) = f(x)$.

If X is connected (resp. simply connected), then there exists at most one (resp. exactly one) map $\tilde{f}: X \to E$ such that $p \circ \tilde{f} = f$ and $\tilde{f}(x) = e$.

Proof. WLOG, $X = B$, $f = \mathrm{id}_B$.

Suppose B connected and $s, s' : B \to E$ are two sections of p s.t. $s(x) = s'(x) = e$.
Let $A \subset B$ be the locus where $s = s'$. Since $x \in A$ and B is connected, it suffices to show that A is open and closed.



- A is open: let $b \in A$, $\exists$ open nhood U of b s.t. $\begin{tikzcd} p^{-1}(U) \arrow[r, "\cong"] \arrow[dr] & I \times U \arrow[d] \\ & U \end{tikzcd}$ where I is discrete.

  Let $i \in I$ such that $s(b) = (i, b) = s'(b)$

  Then $s = s'$ on $s^{-1}(\{i\} \times U) \cap (s')^{-1}(\{i\} \times U) \ni b$

  because they are both sections of $p|_{\{i\} \times U}$, which is an isomorphism.

- $B \setminus A$ is open: let $b \in B \setminus A$, $\exists$ U as above,

$$s(b) = (i, b) \quad s'(b) = (i', b) \quad i \neq i'$$

  then s and s' differ on all $s^{-1}(\{i\} \times U) \cap (s')^{-1}(\{i'\} \times U) \ni b$.

Suppose B simply connected. Then p is trivial, so a section exists. $\square$.

Lemma (criterion for triviality of a covering).

Let $p : E \to B$ is a covering where B is connected. Then p is trivial if and only if for every $e \in E$ there exists a section s of p such that $s(p(e)) = e$.

Proof $\Rightarrow$ clear.

$\Leftarrow$ Let $b \in B$. For each $e \in p^{-1}(b)$, let $s_e : B \to E$ be a section such that $s_e(b) = e$.

A section of p is both open and closed (locally p is the projection $I \times U \to U$ where I is discrete).

B connected $\Rightarrow$ $s_e(B) \cap s_{e'}(B) = \emptyset$

$$E = \bigcup_{e \in p^{-1}(b)} s_e(B)$$

$$\Rightarrow \quad \begin{tikzcd} \coprod\limits_{e \in p^{-1}(b)} B \arrow[rr, "\cong", "(s_e)"'] \arrow[dr] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

$\square$

Proposition

every nhood contains an open connected nhood.

1) Let X be a connected and locally connected space such that for every connected opens $U, V \subset X$, $U \cap V$ is either empty or connected. Then X is simply connected.

2) If X, Y are simply connected and X is locally connected, then $X \times Y$ is simply connected.

Corollary $\mathbb{R}^n$ and I^n are simply connected for all $n \geq 0$.

Proof. 1) Let $E \xrightarrow{p} X$ be a covering and $e \in E$, $x = p(e)$. We want show that there exists a section s of p such that $s(x) = e$.

$$\begin{tikzcd} & E \arrow[d, "p"] \\ U \arrow[ur, dashed, "s_U"] \arrow[r, phantom, "\subset"] & X \end{tikzcd}$$

If $x \in U$ is a connected open nhood, there exists at most one section $s_U : U \to E$ such that $s_U(x) = e$.

If s_U & s_V exist for U, V connected, then since $U \cap V$ is connected we have $s_U|_{U\cap V} = s_V|_{U \cap V}$.

Let Y be the union of all U's such that s_U exists. Then the s_U's induce a section $s : Y \to E$ of p over Y.

It remains to show that $Y = X$. Since X is connected it suffices to show that Y is closed in X. Let $y \in \overline{Y}$. There is a connected open nhood V of y s.t. p is trivial over V. Let $z \in Y \cap V$, and let $s_V : V \to E$ be a section such that $s_V(z) = s(z)$.

Since $Y \cap V$ is connected $\Rightarrow s_V|_{Y\cap V} = s|_{Y \cap V}$

$\Rightarrow$ we get a section over $Y \cup V$

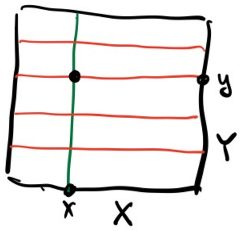
$\Rightarrow V \subset Y \Rightarrow \overline{Y} = Y$.

2) Let $E \xrightarrow{p} X \times Y$, $e \in E$, $p(e) = (x,y)$. We want a section s of p such that $s(x,y) = e$.

Since Y is simply connected, $\exists!$ section $\sigma : \{x\} \times Y \to E$ s.t. $\sigma(x,y) = e$.

Since X is simply connected, $\exists!$ sections $s_z : X \times \{z\} \to E$ s.t. $s_z(x,z) = \sigma(x,z)$ for all $z \in Y$.

Define $s : X \times Y \longrightarrow E$. We need to show that s
$(w, z) \longmapsto s_z(w, z)$ is continuous.



Continuity of s:

- at (x, z), $z \in Y$:

  Let $U \times V$ be an open nbhood of (x, z) over which p is trivial, with U connected.

  $\exists\ \sigma : U \times V \longrightarrow E$ section s.t. $\sigma(x, z) = s(x, z)$

  $s|_{\{x\} \times V}$ is continuous $\Rightarrow$ $s = \sigma$ on $\{x\} \times V'$ for some $z \in V' \subset V$

  (namely choose V' so that both s and σ send $\{x\} \times V'$ to the same sheet)

  For $z' \in V'$, $U \times \{z'\}$ is connected & $s(x, z') = \sigma(x, z')$

  $\Rightarrow$ $s = \sigma$ on $U \times \{z'\}$

  $\Rightarrow$ $s = \sigma$ on $U \times V'$, so s is continuous at (x, z)

- General case. For $z \in Y$, let

$$C(z) = \{ w \in X \mid s \text{ is continuous at } (w, z) \}$$

We know: • $C(z)$ is open

• $x \in C(z)$

Since X is connected, it remains to show $C(z)$ is closed in X.

Let $w \in \overline{C(z)}$. Choose an open nbhood $U \times V$ of (w, z) and a section $\sigma : U \times V \longrightarrow E$ s.t. $\sigma(w, z) = s(w, z)$.

$s|_{U \times \{z\}}$ continuous $\Rightarrow$ $s = \sigma$ on $U' \times \{z\}$ for some $w \in U' \subset U$
↳ connected.

Let $w' \in U' \cap C(z)$

s continuous at (w', z) $\Rightarrow$ $\exists\ V' \subset V$ nbhood of z s.t. $s|_{\{w'\} \times V'}$ is continuous

$\Rightarrow$ $s = \sigma$ on $\{w'\} \times V''$ for some $z \in V'' \subset V'$.

For $z' \in V''$, $U' \times \{z'\}$ is connected & $\sigma(w', z') = s(w', z')$

$\Rightarrow$ $s = \sigma$ on $U' \times \{z'\}$.

$\Rightarrow$ $s = \sigma$ on $U' \times V''$, so s is continuous at (w, z)

$\Rightarrow$ $w \in C(z)$, so $C(z) = \overline{C(z)}$. $\square$