Monodromy

Let $p: E \to B$ be a covering. We write $E_b = p^{-1}(b)$ for any $b \in B$.

If α is a path from b to b' in B and $e \in E_b$, there is a unique $\tilde{\alpha}_e : I \to E$ lifting α such that $\tilde{\alpha}_e(0) = e$ (since I is simply connected).

We define a map $\alpha_* : E_b \to E_{b'}$ by $\alpha_*(e) = \tilde{\alpha}_e(1)$.

Let $H: I \times I \to B$ be a path-homotopy from α to α'. Since $I \times I$ is simply connected, $\exists!\ \tilde{H}: I \times I \to E$ lifting H such that $\tilde{H}(0,0) = e$.

$\Rightarrow$ $\tilde{H}(-,0)$ lifts $H(-,0) = \alpha \Rightarrow \tilde{H}(-,0) = \tilde{\alpha}_e$

$\tilde{H}(-,1)$ lifts $H(-,1) = \alpha' \Rightarrow \tilde{H}(-,1) = \tilde{\alpha}'_e$

$\tilde{H}(0,-)$ lifts $H(0,-)$ which is constant at b, so $\tilde{H}(0,-)$ is constant.

$\tilde{H}(1,-)$ lifts a constant path, hence it is constant.

$\Rightarrow$ $\tilde{H}$ is a path-homotopy from $\tilde{\alpha}_e$ to $\tilde{\alpha}'_e$.

In particular, $\tilde{\alpha}_e(1) = \tilde{\alpha}'_e(1)$.

Hence $\alpha_* = \alpha'_* : E_b \to E_{b'}$.

Moreover:
- $(c_b)_* = id_{E_b}$
- $(\alpha * \beta)_* = \beta_* \circ \alpha_* : E_{\alpha(0)} \to E_{\beta(1)}$ because $\tilde{\alpha}_e * \tilde{\beta}_{\tilde{\alpha}_e(1)}$ lifts $\alpha * \beta$.

Hence, we have defined a functor

$$\begin{aligned} fib(p): \Pi_1(B) &\longrightarrow \mathbf{Set} \\ b &\longmapsto E_b = fib_b(p) \\ [\alpha]_p &\longmapsto \alpha_* \end{aligned}$$

Now suppose
$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$
is a morphism in Cov_B. The square

$$\begin{tikzcd} e \in E_b \arrow[r, "f"] \arrow[d, "\alpha_*"'] & E'_b \arrow[d, "\alpha_*"] \\ E_{b'} \arrow[r, "f"'] & E'_{b'} \end{tikzcd}$$

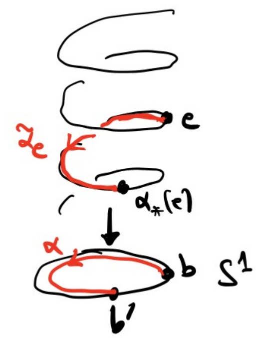
commutes, i.e.: $\tilde{\alpha}_{f(e)}(1) = f(\tilde{\alpha}_e(1))$ since $f \circ \tilde{\alpha}_e$ lifts α.

$\Rightarrow$ f induces a natural transformation

$$\mathrm{fib}(p) \Rightarrow \mathrm{fib}(p')$$

Finally, we get a functor

$$\mathrm{fib} : \mathrm{Cov}_B \longrightarrow \mathrm{Fun}(\Pi_1(B), \mathrm{Set})$$
$$p \longmapsto \mathrm{fib}(p)$$



In particular, the group $\pi_1(B, b)$ acts on the fiber E_b.
This action is called the <u>monodromy</u>.
It has the following features:

- two points $e, e' \in E_b$ are in the same orbit iff they are in the same path-connected component of E.
- the map $p_* : \pi_1(E, e) \to \pi_1(B, b)$ is <u>injective</u> and its image is

$$\{ [\alpha]_p \in \pi_1(B,b) \mid \tilde{\alpha}_e \text{ is a loop} \} = \{ [\alpha]_p \mid \alpha_*(e) = e \} = \mathrm{Stab}_{\pi_1(B,b)}(e \in E_b)$$

<u>Remark</u> One can summarize these observations by an exact sequence:

$$1 \longrightarrow \pi_1(E,e) \xrightarrow{p_*} \pi_1(B,b) \longrightarrow \underset{\substack{\circlearrowleft \\ \pi_1(B,b)}}{E_b} \longrightarrow \pi_0 E \xrightarrow{p_*} \pi_0 B \qquad \left(\pi_0 = \text{path-connected components}\right)$$

$$[\alpha]_p \longmapsto \alpha_*(e)$$

<u>Theorem</u> (Classification of coverings)

Suppose B satisfies:

(+) B is locally path-connected and every $b \in B$ admits a neighborhood U such that $\pi_1(U, b) \to \pi_1(B, b)$ is trivial.

(e.g. B is locally simply path-connected). Then the functor

$$\mathrm{fib} : \mathrm{Cov}_B \longrightarrow \mathrm{Fun}(\Pi_1(B), \mathrm{Set})$$

is an equivalence of categories.

<u>Corollary</u> If B satisfies (+), then B is simply connected iff it is simply path-connected.

In particular, suppose B satisfies $(\dagger)$ and is connected. Let $b \in B$, $G = \pi_1(B,b)$.

Then
$$\begin{aligned} \mathrm{fib}_b : \mathrm{Cov}_B &\xrightarrow{\simeq} \mathrm{Set}_G \\ E &\longmapsto E_b \circlearrowleft G \end{aligned}$$

$$\pi_0 E \cong \text{orbits of the action on } E_b$$
$$\pi_1(E,e) \cong \mathrm{Stab}_G(e \in E_b)$$

E connected $\Leftrightarrow$ G acts transitively on E_b $(\cong G/H)$

E Galois $\Updownarrow$ $\mathrm{Aut}_B(E) \overset{\text{trans.}}{\circlearrowright} E_b$ $\Leftrightarrow$ $\mathrm{Aut}_G(G/H) \overset{\text{trans}}{\circlearrowright} G/H$ $\Updownarrow$ H normal subgroup of G

<u>Proof of the classification theorem</u>

B locally path-connected $\Rightarrow$ $B = \coprod_{i \in I} B_i$ where B_i is path-connected.

$$\mathrm{Cov}_B \simeq \prod_{i \in I} \mathrm{Cov}_{B_i}$$

$$\mathrm{Fun}(\Pi_1(B), \mathrm{Set}) \simeq \prod_{i \in I} \mathrm{Fun}(\Pi_1(B_i), \mathrm{Set})$$

$\Rightarrow$ WLOG we assume B path-connected. Choose $b \in B$. We must show that

$$\mathrm{fib}_b : \mathrm{Cov}_B \longrightarrow \mathrm{Set}_{\pi_1(B,b)} \quad \text{is an equivalence.}$$
$$(p: E \to B) \longmapsto E_b \circlearrowleft \pi_1(B,b).$$

<u>fib_b is fully faithful</u> [this only uses that B is locally path-connected]

Special case:
$$\mathrm{Hom}_B(B,E) \xrightarrow{ev_b} \mathrm{Hom}_{\pi_1(B,b)}(*, E_b) = E_b^{\pi_1(B,b)}$$
$$s \longmapsto s(b)$$

the fixed points of the action.

Recall that any section s is both open and closed, and if $s \neq s'$ then $s(B) \cap s'(B) = \emptyset$.

Hence, if $s(b) = s'(b)$, then $s = s'$ $\Rightarrow$ ev_b is injective.

Suppose $e \in E_b$ is such that $\alpha_*(e) = e$ for all $[\alpha]_p \in \pi_1(B,b)$.

Define $s: B \to E$ by $s(x) = \gamma_*(e)$ where γ is any path from b to x in B.

- s is well-defined: if γ and γ' are two paths from b to x then
$$(\gamma * \bar{\gamma}')_*(e) = e \quad \text{so} \quad \gamma_*(e) = \gamma'_*(e).$$

- s is continuous: let U be a path-connected open neighborhood of $x \in B$ over which $p^{-1}(U) \cong F \times U$ $\searrow U \swarrow$

For any $y \in U$, if γ is a path from x to y in U, then
$s(y) = \gamma_* s(x) = \tilde{\gamma}_{s(x)}(1) \Rightarrow s(x)$ & $s(y)$ are in the same
path-connected component of $p^{-1}(U) \cong F \times U$
$\Rightarrow s|_U : U \longrightarrow F \times U$ is continuous.

$\Rightarrow$ ev_b is surjective.

General case: $\quad \mathrm{Hom}_B(E, E') \xrightarrow{\cong ?} \mathrm{Hom}_{\pi_1(B,b)}(E_b, E'_b)$

Both sides transform $\coprod$ of E's into products, so we can assume E connected.

Choose $e \in E_b$

$$\begin{tikzcd}
 & E \times_B E' \arrow[dl] \arrow[dr] & \\
E \arrow[ur, dashed] \arrow[rr, dashed] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\
 & B &
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto] & (id, f) \\
\mathrm{Hom}_B(E, E') \arrow[r, "\cong"] \arrow[d, "fib_b"'] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{Hom}_E(E, E \times_B E') \arrow[d, "\cong"', "ev_e \text{ (special case)}"] \\
\mathrm{Hom}_{\pi_1(B,b)}(E_b, E'_b) \arrow[r, "\cong"] & (E \times_B E')_e^{\pi_1(E,e)} = (E'_b)^{\pi_1(E,e)} \\
f \arrow[r, mapsto] & f(e)
\end{tikzcd}$$

since $\pi_1(B,b)$ acts transitively on E_b
and the stabilizer at e is $\pi_1(E,e)$.

$\Rightarrow$ the left vertical map is a bijection.