REGENSBURG RESEARCH SEMINAR WS 2020/21

HERMITIAN K-THEORY FOR STABLE ∞-CATEGORIES

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Introduction

The goal of this seminar is to study the foundations of Hermitian K-theory recently developed by Calmès, Dotto, Harpaz, Hebestreit, Land, Moi, Nardin, Nikolaus, and Steimle in [CDH+20a, CDH+20b, CDH+20c]. They associate a genuine C_2-spectrum $\mathrm{KR}(\mathcal{C}, Ϙ)$ to any stable ∞-category $\mathcal{C}$ with a non-degenerate quadratic functor $Ϙ\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Sp}$, whose underlying spectrum is the algebraic K-theory spectrum $\mathrm{K}(\mathcal{C})$, whose fixed points is the Grothendieck–Witt theory spectrum $\mathrm{GW}(\mathcal{C}, Ϙ)$, and whose geometric fixed points is the L-theory spectrum $\mathrm{L}(\mathcal{C}, Ϙ)$ as defined by Lurie [Lur11]. Their construction has good formal properties whether or not 2 is invertible and leads to the computation of the Hermitian K-theory of the integers.

In this seminar we will cover the general theory of Poincaré ∞-categories [CDH+20a] and the construction of Hermitian K-theory [CDH+20b].

1. **Introduction (3.11)**

Part I: Foundations

Numbers refer to [CDH+20a].

2. **Hermitian and Poincaré ∞-categories (10.11)**
   Define quadratic functors (1.1.14), hermitian and Poincaré ∞-categories (1.2.1, 1.2.8). Explain the classification of hermitian structures in terms of a symmetric bilinear part and a linear part (1.3.12, 1.3.13).
3. **Poincaré objects and L-groups (17.11)**
   Define hermitian and Poincaré objects in a hermitian ∞-category (2.1.1, 2.1.3). Define the hyperbolic category (2.2.2) and identify its Poincaré objects (2.2.5). Define the L-groups of and the 0th Grothendieck–Witt group of a Poincaré ∞-category (2.3.11, 2.4.1). Explain the algebraic Thom isomorphism (2.3.20).
4. **Poincaré structures on module categories (24.11)**
   Define modules with involution and the associated quadratic and symmetric hermitian structures $Ϙ^{\mathrm{q}}_M$ and $Ϙ^{\mathrm{s}}_M$ (3.1.1, 3.1.5). Define more generally modules with genuine involution and the associated hermitian structure $Ϙ^{\alpha}_M$ (3.2.2, 3.2.5, 3.2.6). Construct the tower of hermitian structures $Ϙ^{\geq n}_M$ interpolating between $Ϙ^{\mathrm{q}}_M$ and $Ϙ^{\mathrm{s}}_M$ (3.2.7). Prove the periodicity theorems for these hermitian structures (3.4.2, 3.4.10, 3.4.11).
5. **Examples of Poincaré ∞-categories (1.12)**
   Define the universal Poincaré ∞-category (4.1.3). Classify the hermitian structures on $\mathcal{D}^{\mathrm{perf}}(R)$ for R an ordinary associative ring (4.2.12), and identify the hermitian structures $Ϙ^{\geq 0}$, $Ϙ^{\geq 1}$, and $Ϙ^{\geq 2}$ in these terms (4.2.15). Discuss other examples if time permits.

6. **The ∞-category of Poincaré ∞-categories (8.12)**
Define the internal hom between hermitian ∞-categories (6.2.3), and cotensoring and tensoring by ∞-categories (6.3.1, 6.4.1). Show that cotensoring and tensoring by a finite simplicial complex preserves Poincaré ∞-categories (6.6.1). If time permits, discuss the symmetric monoidal structure (5.1.2).

Part II: Cobordism categories and additivity

Numbers refer to [CDH+20b].

7. **Poincaré–Verdier sequences and additive functors (15.12)**
Define Poincaré–Verdier/Karoubi sequences (1.1.1), squares (1.5.1), additive functors, and Verdier/Karoubi-localizing functors (1.5.4). Characterize Poincaré–Verdier inclusions and projections (1.1.5, 1.1.6). Introduce the metabolic fiber sequence (1.2.5). Prove that localizations of rings give rise to Poincaré–Verdier/Karoubi projections (1.4.8).
8. **The hermitian Q-construction and algebraic cobordism categories (22.12)**
Introduce the hermitian Q-construction of a hermitian ∞-category (2.1.1). Show that the hermitian Q-construction is a complete Segal object (2.1.5), that it preserves Poincaré ∞-categories (2.1.6), and that its face maps are split Poincaré–Verdier projections (2.1.7). Define the cobordism ∞-category of an additive functor (2.2.2). Prove the additivity theorem (2.4.1) and the isotropic decomposition theorem (3.2.10).
9. **Structure theory for additive functors (12.1)**
Discuss the delooping theorem for additive functors (3.3.4). Define the spectrification of an additive functor (3.4.3) and prove its universal property (3.4.6) and its connectivity property (3.4.8). Explain the fundamental fiber square (3.6.7) and the formula for bordification via the ρ-construction (3.6.12).
10. **Grothendieck–Witt theory (19.1)**
Define the Grothendieck–Witt space of a Poincaré ∞-category (4.1.1) and explain the Bott–Genauer fiber sequence (4.1.6). Define the Grothendieck–Witt spectrum (4.2.1). Prove Karoubi's fundamental theorem (4.3.3). Define the L-theory space of a Poincaré ∞-category (4.4.1) and prove Lurie's localization theorem for L-theory (4.4.2). Discuss the universal property of L-theory (4.4.12).
11. **The real algebraic K-theory spectrum (26.1)**
Define genuine C_2-equivariant refinements of quadratic functors (I.7.4.17) and the hyperbolic Mackey functor (I.7.4.18). Define the genuine hyperbolization of an additive functor (II.3.7.2) and the real algebraic K-theory spectrum (II.4.5.2). Prove the genuine Karoubi periodicity theorem (II.3.7.7) and its consequences for real algebraic K-theory (II.4.5.4–4.5.6).
12. **Comparison with group completion (2.2)**
Prove the comparison between $\mathrm{GW}(\mathcal{D}^{\mathrm{perf}}(R), \Omega^{\mathrm{gq/gs}})$ and the group completion of the groupoid of non-degenerate quadratic/symmetric forms [HS20].

References

[CDH+20a] B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, M. Land, K. Moi, D. Nardin, T. Nikolaus, and W. Steimle, *Hermitian K-theory for stable ∞-categories I: Foundations*, 2020, arXiv:2009.07223

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[HS20] F. Hebestreit and W. Steimle, *Stable moduli spaces of hermitian forms*, 2020

[Lur11] J. Lurie, *Algebraic L-theory and Surgery*, course notes, 2011, https://www.math.ias.edu/~lurie/287x.html