Real Algebraic K-theory

Winter Semester 2020/2021

Oberseminar: Hermitian K-theory for stable ∞-categories

@ Universität Regensburg

Gabriel Angelini-Knoll
Postdoctoral Researcher
Freie Universität
Berlin

Goal:

1) Give a genuine C_2-equivariant lift of GW

$$\begin{tikzcd} & Sp^{gC_2} \arrow[d] \\ Cat_\infty^p \arrow[ur, "KR"] \arrow[r, "GW"'] & Sp \end{tikzcd}$$

to Sp^{gC_2}. Show that the isotropy separation diagram

$$\begin{tikzcd} GW(\mathcal{C}, Ϙ) \simeq KR(\mathcal{C}, Ϙ)^{C_2} \arrow[r] \arrow[d] & KR(\mathcal{C}, Ϙ)^{\varphi C_2} \simeq L(\mathcal{C}, Ϙ) \arrow[d] \\ K(\mathcal{C})^{hC_2} \simeq KR(\mathcal{C}, Ϙ)^{hC_2} \arrow[r] & KR(\mathcal{C}, Ϙ)^{tC_2} \simeq K(\mathcal{C})^{tC_2} \end{tikzcd}$$

2) Prove the genuine Karoubi periodicity theorem

$$KR(\mathcal{C}, Ϙ^{[-1]}) \simeq S^{\sigma-1} \otimes KR(\mathcal{C}, Ϙ)$$

and discuss its corollaries.

I. Genuine C_2-equivariant lifts of additive functors

<u>Def</u>: A C_2-category is a cocartesian fibration

$$\mathcal{E} \to \mathcal{O}_{C_2}^{op} \quad (\mathcal{O}_{C_2}^{op} \to \mathrm{Cat}_\infty)$$

[Elmendorff's theorem $\mathrm{Fun}(\mathcal{O}_{C_2}^{op}, \mathrm{Top}) \simeq \mathrm{Top}_{C_2}$]

<u>Rmk</u>: A C_2-category is the data of a functor

$$f : \mathcal{E}_* \longrightarrow \mathcal{E}_{C_2}^{\;C_2}$$

$* = C_2/C_2$

$C_2 = C_2/e$

where $\mathcal{E}_{C_2}$ has a C_2-action and f is equivariant with respect to the trivial action on $\mathcal{E}_*$.

$$\rightsquigarrow \mathcal{E}_* \longrightarrow \mathcal{E}_{C_2}^{hC_2}$$

<u>Def</u>. We say a C_2-category admits finite C_2-(co)products if $\mathcal{E}_*$ and $\mathcal{E}_{C_2}$ admit finite (co)products, $f : \mathcal{E}_* \to \mathcal{E}_{C_2}$ preserves them,

$$f : \mathcal{E}_* \underset{\underset{h}{\longleftarrow}}{\overset{\overset{g}{\longleftarrow}}{\underset{\perp}{\overset{\perp}{\longrightarrow}}}} \mathcal{E}_{C_2}$$

1) $x \amalg \sigma x \xrightarrow{\simeq} fg(x)$

2) $fh(x) \xrightarrow{\simeq} x \times \sigma x$.

$$\sigma : \mathcal{E}_{C_2} \to \mathcal{E}_{C_2}$$

Def. We say in addition that a C_2-category is C_2-semiadditive if $\mathcal{E}_*, \mathcal{E}_{C_2}$ are semiadditive and the canonical nat. trans.

$$g(-) \xrightarrow{\simeq} h(-)$$

is an equivalence. [Wirthmüller isomorphism]

Examples:

1) $\underline{\mathrm{Fun}}^{q}(\mathcal{C}) \longrightarrow \mathcal{O}_{C_2}^{op}$

$$\underline{\mathrm{Fun}}^{q}(\mathcal{C})_* = \mathrm{Fun}^{q}(\mathcal{C}) \longrightarrow \mathrm{Fun}^{sb}(\mathcal{C}) = \mathrm{Fun}^{b}(\mathcal{C})^{hC_2} = \underline{\mathrm{Fun}}^{q}(\mathcal{C})_{C_2}$$

$$Ϙ \longmapsto B_Ϙ(-,-)$$

2) $\underline{\mathrm{Cat}}_\infty^{p} \longrightarrow \mathcal{O}_{C_2}^{op}$

$$\mathrm{Cat}_\infty^{p} \longrightarrow \mathrm{Cat}_\infty^{sp} = (\mathrm{Cat}_\infty^{ex})^{hC_2}$$

$$(\mathcal{C}, Ϙ) \longmapsto (\mathcal{C}, D_Ϙ : \mathcal{C} \xrightarrow{\simeq} \mathcal{C}^{op})$$

Rmk: These are both C_2-semiadditive!

Ex:

$$U : \mathrm{Cat}^p_\infty \underset{\perp}{\rightleftarrows} \mathrm{Cat}^{ex}_\infty : \mathrm{Hyp}$$

$$U^{hC_2} : \mathrm{Fun}(BC_2, \mathrm{Cat}^p_\infty) \underset{\perp}{\rightleftarrows} (\mathrm{Cat}^{ex}_\infty)^{hC_2} : \mathrm{Hyp}^{hC_2}$$

$$\overline{\mathrm{Hyp}} : \mathrm{Cat}^p_\infty \to (\mathrm{Cat}^{ex}_\infty)^{hC_2} \xrightarrow{\mathrm{Hyp}^{hC_2}} \mathrm{Fun}(BC_2, \mathrm{Cat}^p_\infty)$$

Rmk:

$$\begin{tikzcd}
\mathrm{Cat}^p_\infty \arrow[dr] & & \mathrm{Cat}^p_\infty \arrow[dr, "U"] & \\
& \mathrm{Cat}^{ex}_\infty \arrow[ur, "\mathrm{Hyp}"] \arrow[rr] \arrow[d, "\mathrm{Hyp}"'] & & (\mathrm{Cat}^{ex}_\infty)^{hC_2} \arrow[d, "\mathrm{Hyp}^{hC_2}"] \\
& \mathrm{Cat}^p_\infty \arrow[rr] & & \mathrm{Fun}(BC_2, \mathrm{Cat}^p_\infty)
\end{tikzcd}$$

$$\begin{aligned}
\mathrm{Cat}^p_\infty &\longrightarrow \mathrm{Fun}(BC_2, \mathrm{Cat}^p_\infty) \\
(\mathcal{C}, Ϙ) &\longmapsto \overline{\mathrm{Hyp}}(\mathrm{Hyp}(\mathcal{C})) \\
&\qquad \cong \\
&\quad C_2 \otimes \mathrm{Hyp}(\mathcal{C})
\end{aligned}$$

Prop. I.7.4.16

Suppose

$$f : \Sigma_* \underset{\perp}{\rightleftarrows} \Sigma_{C_2} : g$$

is a C_2-semiadditive C_2-category.

Then there is a canonical lift

$$\begin{tikzcd} & \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{C_2}), \Sigma_*) \arrow[d] \\ \Sigma_* \arrow[ur] \arrow[r, equal] & \Sigma_* \end{tikzcd}$$

Rmk: Guillou-May theorem $\mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{C_2}), Sp) = Sp^{gC_2}$

$\mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{C_2}), \Sigma_*)$

"Mackey objects in Σ_*".

Pf sketch.

$$\begin{tikzcd} & \mathrm{Fun}^{\mathrm{cocart}}_{/\mathcal{O}^{op}_{C_2}}(\underline{A}^{\mathrm{eff}}(\mathbb{F}_{C_2}), \Sigma) \\ \Sigma_* \arrow[r, "\simeq"] & \mathrm{Fun}^{\oplus}_{C_2}(\underline{A}^{\mathrm{eff}}(\mathbb{F}_{C_2}), \Sigma) \arrow[u, phantom, "\cup"] \arrow[r] & \mathrm{Fun}^\times(\mathrm{Span}(\mathbb{F}_{C_2}), \Sigma_*) \end{tikzcd}$$

Nar 16

basechange along $\{*\} \to \mathcal{O}^{op}_{C_2}$

Cor. I.7.4.17

$$\begin{tikzcd} & \tilde{\varphi} \arrow[dr, bend left] & \\ \varphi \arrow[ur, <->] & & \end{tikzcd}$$

$$\begin{tikzcd} & \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_{C_2}), \mathrm{Fun}^{q}(\mathcal{C})) \arrow[d, "\mathrm{ev}_0"] \\ \mathrm{Fun}^{q}(\mathcal{C}) \arrow[r, equal] & \mathrm{Fun}^{q}(\mathcal{C}) \end{tikzcd}$$

$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_{C_2}), \mathrm{Fun}^{q}(\mathcal{C})) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Fun}^{\times}(\mathrm{Span}\mathbb{F}_{C_2}, Sp))$

$\mathrm{Fun}^{q}(\mathcal{C}) \in \mathrm{Fun}(\mathcal{C}^{op}, Sp^{gC_2})$

[Lewis diagram]

$$\begin{tikzcd} \tilde{\varphi}(x) = \varphi \arrow[d, bend right] \\ \tilde{\varphi}(C_2) = B_{\varphi}(-,-) \arrow[u, bend right] \end{tikzcd}$$

$\mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_{C_2}), \mathrm{Fun}^{q}(\mathcal{C})) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, Sp^{gC_2}]$

$\cap$

$\mathrm{Fun}(\mathrm{Span}(\mathbb{F}_{C_2}), \mathrm{Fun}(\mathcal{C}^{op}, Sp)) \simeq$

$\simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Fun}(\mathrm{Span}(\mathbb{F}_{C_2}), Sp))$

$$\begin{tikzcd} \varphi(-) \arrow[r] \arrow[d] & L_{\varphi}(-) \arrow[d] \\ B_{\varphi}(-,-)^{hC_2} \arrow[r] & B_{\varphi}(-,-)^{tC_2} \end{tikzcd}$$

Cor. I.7.4.18

$$\begin{tikzcd} & \mathrm{Fun}^{\times}(\mathrm{Span}(\mathbb{F}_{C_2}), \mathrm{Cat}_{\infty}^{p}) \arrow[d] \\ \mathrm{Cat}_{\infty}^{p} \arrow[ur, "g\mathrm{Hyp}"] \arrow[r] & \mathrm{Cat}_{\infty}^{p} \end{tikzcd}$$

unit/counit of adjunctions

$U: \mathrm{Cat}_{\infty}^{p} \underset{\perp}{\overset{\leftarrow}{\rightarrow}} \mathrm{Cat}_{\infty}^{ex}: \mathrm{Hyp}$

$$\begin{tikzcd} (g\mathrm{Hyp}(\mathcal{C}, \varphi))(x) = (\mathcal{C}, \varphi) \arrow[d, bend right, "fgt"'] \\ (g\mathrm{Hyp}(\mathcal{C}, \varphi))(C_2) = \mathrm{Hyp}(\mathcal{C}) \arrow[u, bend right, "\mathrm{hyp}"'] \end{tikzcd}$$

Def II.3.7.2

Given an additive functor

$$F : \mathrm{Cat}_\infty^p \longrightarrow \mathrm{Sp}$$

we define

$$\begin{tikzcd}
 & \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_{C_2}), \mathrm{Cat}_\infty^p) \arrow[r, "F"] \arrow[dd, "\mathrm{ev}_*"] & \mathrm{Fun}^\times(\mathrm{Span}(\mathrm{Fin}_{C_2}), \mathrm{Sp}) \arrow[d, phantom, "\parallel"] \\
 & & \mathrm{Sp}^{gC_2} \arrow[d, "\mathrm{ev}_*"] \\
\mathrm{Cat}_\infty^p \arrow[uur, "g\mathrm{Hyp}"] \arrow[uurr, bend left, "F^{g\mathrm{Hyp}}"] \arrow[r, equal] \arrow[rr, bend right, "F"'] & \mathrm{Cat}_\infty^p \arrow[r] & \mathrm{Sp}.
\end{tikzcd}$$

Def:

$$KR(\mathcal{C}, Ϙ) = GW^{g\mathrm{Hyp}}(\mathcal{C}, Ϙ).$$

Consequently, we recover

$$\begin{tikzcd}
\mathcal{F}(\mathcal{C},\Qoppa) \simeq \mathcal{F}^{\mathrm{gHyp}}(\mathcal{C},\Qoppa)^{C_2} \arrow[r] \arrow[d] & \mathcal{F}^{\mathrm{gHyp}}(\mathcal{C},\Qoppa)^{\varphi C_2} \simeq \widetilde{\mathcal{F}}^{\mathrm{bord}}(\mathcal{C},\Qoppa) \arrow[d] \\
\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^{hC_2} \simeq \mathcal{F}^{\mathrm{gHyp}}(\mathcal{C},\Qoppa)^{hC_2} \arrow[r] & \mathcal{F}^{\mathrm{gHyp}}(\mathcal{C},\Qoppa)^{tC_2} \\
\mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^{hC_2} \simeq \mathcal{F}^{\mathrm{hyp}}(\mathcal{C},\Qoppa)^{hC_2} & \mathcal{F}^{\mathrm{gHyp}}(\mathcal{C},\Qoppa)^{tC_2} \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^{tC_2} \simeq \mathcal{F}^{\mathrm{hyp}}(\mathcal{C},\Qoppa)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{GW}(\mathcal{C},\Qoppa) \simeq \mathrm{KR}(\mathcal{C},\Qoppa)^{C_2} \arrow[r] \arrow[d] & \mathrm{KR}(\mathcal{C},\Qoppa)^{\varphi C_2} \simeq \mathrm{L}(\mathcal{C},\Qoppa) \arrow[d] \\
\mathrm{K}(\mathcal{C})^{hC_2} \simeq \mathrm{KR}(\mathcal{C},\Qoppa)^{hC_2} \arrow[r] & \mathrm{KR}(\mathcal{C},\Qoppa)^{tC_2} \simeq \mathrm{K}(\mathcal{C})^{tC_2}
\end{tikzcd}$$

II. Genuine Karoubi periodicity

Thm II 3.7.7 (Genuine Karoubi periodicity)

Let $F : \mathrm{Cat}^{p}_{\infty} \to \mathrm{Sp}$ be an additive functor, then there is a natural equivalence of genuine C_2-spectra

$$F^{gHyp}(\mathcal{C}, Ϙ^{[-1]}) \simeq \mathbb{S}^{\sigma-1} \otimes F^{gHyp}(\mathcal{C}, Ϙ).$$

Pf.

Claim:

$$\begin{tikzcd}
F^{gHyp}(\mathrm{Met}(\mathcal{C}, Ϙ)) \arrow[r] \arrow[d, "\simeq"] & F^{gHyp}(\mathcal{C}, Ϙ) \arrow[r] \arrow[d, equal] & \mathbb{S}^{1} \otimes F^{gHyp}(\mathcal{C}, Ϙ^{[-1]}) \arrow[d, "\simeq"] \\
C_2 \otimes F^{gHyp}(\mathcal{C}, Ϙ) \arrow[r] & F^{gHyp}(\mathcal{C}, Ϙ) \arrow[r] & \mathbb{S}^{\sigma} \otimes F^{gHyp}(\mathcal{C}, Ϙ)
\end{tikzcd}$$

$$\| $$

$$(\mathbb{S}[C_2] \to \mathbb{S} \to \mathbb{S}^{\sigma}) \otimes F^{gHyp}(\mathcal{C}, Ϙ) \qquad \square$$

Proof of claim (Lemma II 3.7.6)

1) Recall that $\overline{\mathrm{Hyp}}(\mathrm{Hyp}(\mathcal{C})) \simeq C_2 \otimes \mathrm{Hyp}(\mathcal{C})$

2) $F^{gHyp}(\mathrm{Met}(\mathcal{C}, Ϙ)) \simeq F^{gHyp}(\mathrm{Hyp}(\mathcal{C}))$

3) Suffices to show that there is an equivalence of genuine C_2-spectra

$$\tilde{\mathcal{F}}^{gHyp}(\mathrm{Hyp}(\mathcal{C})) \underset{C_2}{\simeq} C_2 \otimes \tilde{\mathcal{F}}^{gHyp}(\mathcal{C}, \text{♀}).$$

On underlying spectra with C_2-action

$$\begin{aligned} u\tilde{\mathcal{F}}^{gHyp}(\mathrm{Hyp}(\mathcal{C})) &\simeq \tilde{\mathcal{F}}^{hyp}(\mathrm{Hyp}(\mathcal{C})) \\ &\simeq \tilde{\mathcal{F}}(\overline{\mathrm{Hyp}}(\mathrm{Hyp}(\mathcal{C}))) \\ &\simeq \tilde{\mathcal{F}}((C_2 \otimes \mathrm{Hyp}(\mathcal{C})) \\ &\simeq C_2 \otimes \tilde{\mathcal{F}}(\mathrm{Hyp}(\mathcal{C})) \\ &\simeq C_2 \otimes \tilde{\mathcal{F}}^{hyp}(\mathcal{C}, \text{♀}) \\ &\simeq u(C_2 \otimes \tilde{\mathcal{F}}^{gHyp}(\mathcal{C}, \text{♀})). \end{aligned}$$

On geometric fixed points

$$\widetilde{\mathcal{F}}^{gHyp}(\mathrm{Hyp}(\mathcal{C}))^{\varphi C_2} \simeq \widetilde{\mathcal{F}}^{bord}(\mathrm{Hyp}(\mathcal{C}))$$
$$\simeq 0 \text{ by bordism invariance.}$$

$$\left(C_2 \otimes \widetilde{\mathcal{F}}^{gHyp}(\mathcal{C}, Ϙ)\right)^{\varphi C_2}$$
$$\simeq \mathbb{S}[C_2]^{\varphi C_2} \otimes \widetilde{\mathcal{F}}^{gHyp}(\mathcal{C}, Ϙ)^{\varphi C_2}$$
$$\simeq \mathbb{S}[C_2]^{\varphi C_2} \otimes \widetilde{\mathcal{F}}^{bord}(\mathcal{C}, Ϙ)$$
$$\simeq 0 \qquad C_2^{C_2} \simeq \emptyset \qquad \square$$

<u>Cor.</u> II 4.5.3

$$KR(\mathcal{C}, Ϙ^{[1]}) \simeq \mathbb{S}^{1-\sigma} \otimes KR(\mathcal{C}, Ϙ)$$

<u>Rmk:</u>

$$KR(\mathcal{C}, Ϙ)^{\varphi C_2} \quad \text{colim}_d \, \mathbb{S}^d \otimes GW(\mathcal{C}, Ϙ^{[-d]}) = GW^{bord}(\mathcal{C}, Ϙ)$$

$$KR(\mathcal{C}, Ϙ) = \left(\text{colim}_d \, \mathbb{S}^{d\sigma} \otimes KR(\mathcal{C}, Ϙ)\right)^{C_2} \simeq \left(\text{colim}_d \, \mathbb{S}^d \otimes KR(\mathcal{C}, Ϙ^{[-d]})\right)^{C_2}$$

<u>Notation</u>: From now on let R be a complex oriented E_1-ring, M an invertible module with involution over R, $c \subseteq K_0(R)$ a subgroup closed under the involution induced by M.

<u>Cor</u>. II 4.5.4 Let $r \in \{q, s\}$

$$KR(\mathrm{Mod}_R^c, Ϙ^r_{-M}) \simeq S^{2-2\sigma} \otimes KR(\mathrm{Mod}_R^c, Ϙ^r_M)$$

if R is connective,

$$KR(\mathrm{Mod}_R^c, Ϙ^{\geq m+1}_{-M}) \simeq S^{2-2\sigma} \otimes KR(\mathrm{Mod}_R^c, Ϙ^{\geq m}_M)$$

<u>Proof</u>. Note the equivalence of Thm II 3.7.7 is natural and by B. Shin's talk

$$(Ϙ^r_{-M})^{[2]} \simeq Ϙ^r_M \qquad r \in \{q, s\}$$

R connective $\quad (Ϙ^{\geq m+1}_{-M})^{[2]} \simeq Ϙ^{\geq m}_M$. $\square$

$$\Rightarrow \quad (\text{Ϙ}^{r}_{M})^{[4]} \simeq \text{Ϙ}^{r}_{M} \qquad r \in \{q, s\}$$

R connective

$$\text{Ϙ}^{\geq 2}_{M} = \text{Ϙ}^{gq}_{M} \quad \left(\text{Ϙ}^{\geq 1}_{M} = \text{Ϙ}^{ge}_{M}\right)$$

$$\text{Ϙ}^{\geq 0}_{M} = \text{Ϙ}^{gs}_{M}$$

Cor. II 4.5.5 (Karoubi periodicity)

$r \in \{q, s\}$

$KR(\mathrm{Mod}^{c}_{R}, \text{Ϙ}^{r}_{M})$ is $(4-4\sigma)$-periodic

If R real orientable, $M = -M$

$KR(\mathrm{Mod}^{c}_{R}, \text{Ϙ}^{r}_{M})$ is $(2-2\sigma)$-periodic

If R is connective,

$$KR(\mathrm{Mod}^{c}_{R}, \text{Ϙ}^{gq}_{M}) \simeq \mathbb{S}^{4-4\sigma} \otimes KR(\mathrm{Mod}^{c}_{R}, \text{Ϙ}^{gs}_{M})$$

$$(\mathbb{S}^{\sigma} \otimes X)^{C_2} = \mathrm{cof}(X \to X^{C_2})$$

Cor. II 4.5.6 (Ranicki periodicity) $r \in \{q, s\}$

1) $L(\mathrm{Mod}^c_R, \mathring{I}^r_{-M}) \simeq \mathbb{S}^2 \otimes L(\mathrm{Mod}^c_R, \mathring{I}^r_M)$

2) $L(\mathrm{Mod}^c_R, \mathring{I}^r_M)$ is 4-periodic

and 2-periodic if R is real orientable.

3) $L(\mathrm{Mod}^c_R, \mathring{I}^{gq}_M) \simeq \mathbb{S}^4 \otimes L(\mathrm{Mod}^c_R, \mathring{I}^{gs})$.

Pf.

$$\left(\mathbb{S}^{2k-2\sigma} \otimes KR(\mathcal{C}, \mathring{I})\right)^{\varphi C_2}$$

$$\simeq \left(\mathbb{S}^{2k-2\sigma}\right)^{\varphi C_2} \otimes KR(\mathcal{C}, \mathring{I})^{\varphi C_2}$$

$$\simeq \mathbb{S}^{2k} \otimes L(\mathcal{C}, \mathring{I}) \qquad \square$$

$$(\mathbb{S}^0)^{\varphi C_2} \to (\mathbb{S}^\sigma)^{\varphi C_2}$$

$\simeq$