Poincaré Structures on Module Categories

Plan:
1) Modules w/ Involution
2) Modules w/ Genuine Involution
3) Periodicity.

Notation: k a E_∞ ring spectrum, $\mathrm{Alg}_{E_1} = \mathrm{Alg}_{E_1}(\mathrm{Mod}_k)$

$A \in \mathrm{Alg}_{E_1}$, Mod_A = Left A-modules

Mod_A^f = f.p.

Mod_A^ω = compact

§1. Modules w/ Involution

$$\otimes_k : \mathrm{Mod}_k \times \mathrm{Mod}_k \to \mathrm{Mod}_k$$

$$\mathrm{Mod}_A \times \mathrm{Mod}_B \to \mathrm{Mod}_{A \otimes_k B}, \quad A, B \in \mathrm{Alg}_{E_1}$$

if $A = B$, this is C_2-equivariant

construction: for $M \in \mathrm{Mod}_{A \otimes_k A}$, define

$$B_M : \mathrm{Mod}_A^\omega \times \mathrm{Mod}_A^\omega \to \mathrm{Sp} \quad \text{by}$$

$$B_M(X, Y) = \hom_{A \otimes A}(X \otimes Y, M)$$

$$\rightsquigarrow B_{(-)} : \mathrm{Mod}_{A \otimes A} \to \mathrm{Fun}^b(\mathrm{Mod}_A^\omega)$$

C_2-equivariant.

Def'n: A <u>module w/ involution</u> over A is an object of $\left(\mathrm{Mod}_{A\otimes A}\right)^{hC_2}$

Note: $M \in \mathrm{Mod}_{A\otimes A}^{hC_2} \rightsquigarrow B_M \in \mathrm{Fun}^s(\mathrm{Mod}_A^\omega)$

Perspective $M \in \mathrm{Mod}_{A\otimes A} \rightsquigarrow M \in \mathrm{Sp}$ w/ 2 actions of A

$$A \longrightarrow \hom_A(M, M) \qquad (*)$$

if $M \in \mathrm{Mod}_{A\otimes A}^{hC_2}$, then "first" & "second" are interchangeable.

Def'n: An A-mod w/ involution M is <u>invertible</u> if
1) $M \in \mathrm{Mod}_A$ is compact
2) the map $(*)$ is an equivalence.

Proposition: $A \in \mathrm{Alg}_{E_1}$, $M \in (\mathrm{Mod}_{A\otimes A})^{hC_2}$
then $B_M \in \mathrm{Fun}^s(\mathrm{Mod}_A^\omega)$ is nondegenerate iff $M \in \mathrm{Mod}_A^\omega$. In this case, $D(X) = \hom_A(X, M)$.
Moreover, in this case, B_M is perfect iff M is invertible.

Rmk: still true if $\mathrm{Mod}_A^\omega \mapsto \mathrm{Mod}_A^f$

pf: exercise $\square$

Def'n: $C_2 \curvearrowright \mathrm{Alg}_{E_1}$ $\quad A \mapsto A^{op}$
An algebra w/ anti-involution is an object of $\mathrm{Alg}_{E_1}^{hC_2}$

Ex: $A \in \mathrm{Alg}_{E_1}^{hC_2}$, we can view $A \in \mathrm{Mod}_{A\otimes A}^{hC_2}$
via $(a\otimes b)x = axb$

Ex: $A \in \mathrm{Alg}_{E_1}^{hC_2}$, we can view $A \in \mathrm{Mod}_{A\otimes A}^{hC_2}$
via $(a\otimes b)x = ax\bar{b}$

e.g.) H(ordinary R w/ antiinvolution)
E_∞ algebras w/ involution
Group algebras

by above criterion, A is invertible

Def'n: In general, we'll say $M \in \mathrm{Mod}_{A\otimes A}^{hC_2}$ comes from an anti-involution of A if it is equivalent to the above ex.

warning: $A \in \mathrm{Alg}_{E_1}$, we can give it a module involution that does not come from anti-involution. (ex) Wall anti-structure)

Proposition $A \in \mathrm{Alg}_{E_1}$, $M \in \mathrm{Mod}_{A\otimes A}^{hC_2}$
M comes from an anti-involution on A iff there is a C_2-equivariant map $u : \mathbb{S} \to M$ such that the induced A-linear map $A \to M$ is an equivalence.

$$A \xrightarrow{\sim} \hom_A(M,M) \simeq \hom_A(A,A) \simeq A^{op}$$

pf sketch: ($\Rightarrow$) immediate.

($\Leftarrow$) $u : \mathbb{S} \to M \rightsquigarrow m_0 \in M$ w/ $\overline{m_0} = m$

Consider

$$A \longrightarrow \hom_A(M,M)$$

pf sketch: ($\Rightarrow$) immediate.

($\Leftarrow$) $u: \mathbb{S} \to M \rightsquigarrow m_0 \in M$ w/ $\overline{m_0} = m$

Consider

$$\begin{tikzcd} x \arrow[d, mapsto] & A \arrow[r] \arrow[d, "\wr"] \arrow[dr, phantom, "\circlearrowleft"] & \hom_A(M,M) \arrow[d, "\wr"] & f \arrow[d, mapsto] \\ (x\otimes 1)m_0 & M \arrow[r, "\sim", "inv"'] & M & f(m_0) \end{tikzcd}$$

$x \longmapsto (m \mapsto (1\otimes x)m)$

hypothesis $\Rightarrow$ vertical maps are equivalences.

$\rightsquigarrow$ M is invertible

$$A \to \hom_A(M,M) \simeq \hom_A(A,A) \simeq A^{op}$$

$\rightsquigarrow$ A has anti-involution $\square$

Def'n: $M \in \mathrm{Mod}_{A\otimes A}^{hC_2}$

$$\text{Ϙ}^q_M(X) = B_M(X)_{hC_2} \qquad \text{Ϙ}^s_M(X) = B_M(X)^{hC_2}$$

Rmk: there are many more Ϙ w/ $B_\text{Ϙ} = B_M$. need to look at ...

§2. Modules w/ Genuine Involution

Recall: $X \in Sp \rightsquigarrow X \to (X \otimes_{\mathbb{S}} X)^{tC_2}$

if $X \in \mathrm{Mod}_k \rightsquigarrow X \to (X\otimes_{\mathbb{S}} X)^{tC_2} \to (X \otimes_k X)^{tC_2}$ $(*)$

$\uparrow$ $\mathrm{Mod}_{k^{tC_2}}$ "k-linear Tate diag"

if $X = k \rightsquigarrow k \to k^{tC_2}$

$\Rightarrow (X \otimes_k X)^{tC_2} \in \mathrm{Mod}_k$

& $(*)$ is k-linear

Recall: $X \in Sp \rightsquigarrow X \to (X \otimes_S X)^{tC_2}$

if $X \in Mod_k \rightsquigarrow X \to (X \otimes_S X)^{tC_2} \to (X \otimes_k X)^{tC_2}$ (*)

$\uparrow$ $Mod_{k^{tC_2}}$ $\qquad$ $\uparrow$ "k-linear Tate diag"

if $X = k \rightsquigarrow k \to k^{tC_2} \xleftarrow{} $ Tate Frob

$\Rightarrow (X \otimes_k X)^{tC_2} \in Mod_k$

& (*) is k-linear

warning: Tate Frob is not $k \to k^{hC_2} \to k^{tC_2}$

Def'n: An A-mod w/ Gen involution consists of

- $M \in (Mod_{A \otimes_k A})^{hC_2}$
- $N \in Mod_A$
- $\alpha : N \to M^{tC_2}$ an A-linear map.
  $\circlearrowleft A \to (A \otimes_k A)^{tC_2}$

Lemma: for $M \in Mod_{A\otimes A}^{hC_2}$, $X \in Mod_A^{\omega}$, then

$$hom_{A\otimes A}(X \otimes X, M)^{tC_2} \simeq hom_A(X, M^{tC_2})$$

pf: Note

$$(Mod_A^{\omega})^{op} \longrightarrow Sp, \quad X \mapsto hom_{A\otimes A}(X \otimes X, M)^{tC_2}$$

is an exact functor, so by Morita theory, this equiv $hom_A(-, N)$. Evaluating at $X = A$. $\square$

Lemma: for $M \in \mathrm{Mod}_{A\otimes A}^{hC_2}$, $X \in \mathrm{Mod}_A^{\omega}$, then

$$\hom_{A\otimes A}(X\otimes X, M)^{tC_2} \simeq \hom_A(X, M^{tC_2})$$

Construction: (M, N, α) A-mod w/ gen inv

Define

$$\begin{tikzcd} Ϙ_M^\alpha(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \hom_A(X, N) \arrow[d, "\alpha"] \\ Ϙ_M^s(X) \arrow[r] & \hom_A(X, M^{tC_2}) \end{tikzcd}$$

$$Ϙ_M^s(X) = \hom_{A\otimes A}(X\otimes X, M)^{hC_2}$$

Note: $Ϙ_M^\alpha$ has Bilinear part is B_M

$$Ϙ_M^q \longrightarrow Ϙ_M^\alpha \longrightarrow Ϙ_M^s$$

example: $\alpha = (0 \to M^{tC_2}) \rightsquigarrow Ϙ_M^\alpha = Ϙ_M^q$

$\alpha = (M^{tC_2} \xrightarrow{id} M^{tC_2}) \rightsquigarrow Ϙ_M^\alpha = Ϙ_M^s$

examples: A is connective

($\rightsquigarrow$ truncations of A-modules are A-modules)

$M \in \mathrm{Mod}_{A\otimes A}^{hC_2}$, $m \in \mathbb{Z}$

$$\left(M, \tau_{\geq m} M^{tC_2}, \tau_{\geq m} M^{tC_2} \to M^{tC_2}\right) \rightsquigarrow Ϙ_M^{\geq m}$$

<u>examples:</u> A is connective

($\rightsquigarrow$ truncations of A-modules are A-modules)

$M \in \mathrm{Mod}_{A\otimes A}^{hC_2}$, $m \in \mathbb{Z}$

$$\left(M,\ \tau_{\geq m} M^{tC_2},\ \tau_{\geq m} M^{tC_2} \to M^{tC_2}\right) \rightsquigarrow \mathcal{Q}^{\geq m}_M$$

$$\cdots \to \mathcal{Q}^{\geq -1}_M \to \mathcal{Q}^{\geq 0}_M \to \mathcal{Q}^{\geq 1}_M \to \cdots$$

with $\lim = \mathcal{Q}^q_M$, $\operatorname{colim} = \mathcal{Q}^s_M$

<u>Defn:</u>

$$\begin{tikzcd}
\mathrm{Mod}_{NA} \arrow[r] \arrow[d] \arrow[dd, bend right] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathrm{Mod}_A) \arrow[d, "t"] \arrow[dd, bend left] \\
\mathrm{Mod}_{A\otimes A}^{hC_2} \arrow[r, "(-)^{tC_2}"'] \arrow[dd, bend right, "B_{(-)}"'] & \mathrm{Mod}_A \arrow[dd, bend left, "M \mapsto \hom_A(-,M)"] \\
\mathrm{Fun}^q(\mathrm{Mod}_A^\omega) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ar}(\mathrm{Fun}^{ex}((\mathrm{Mod}_A^\omega)^{op}, \mathrm{Sp})) \arrow[d] \\
\mathrm{Fun}^s(\mathrm{Mod}_A^\omega) \arrow[r] & \mathrm{Fun}^{ex}((\mathrm{Mod}_A^\omega)^{op}, \mathrm{Sp})
\end{tikzcd}$$

<u>Note:</u> (bottom square)

<u>Theorem:</u> if $k = \mathbb{S}$, then this is an equivalence of squares. In particular

$$\mathrm{Mod}_{NA} \simeq \mathrm{Fun}^q(\mathrm{Mod}_A^\omega)$$

<u>pf sketch:</u> $\mathrm{Mod}_A \longrightarrow \mathrm{Fun}^{ex}((\mathrm{Mod}_A^\omega)^{op}, \mathrm{Sp})$

$$\mathrm{Mod}_A = \mathrm{Ind}(\mathrm{Mod}_A^\omega) \simeq \mathrm{Fun}^{rex}((\mathrm{Mod}_A^\omega)^{op}, \mathcal{S}) \simeq \mathrm{Fun}^{ex}((\mathrm{Mod}_A^\omega)^{op}, \mathrm{Sp})$$

$\square$

§3. Periodicity

Recall: $\mathcal{I}^{[n]} = \Sigma^n \circ \mathcal{I}$

Observe: $(\mathcal{I}^\alpha_M)^{[n]} = \mathcal{I}^{\Sigma^n \alpha}_{\Sigma^n M}$
$\quad\quad \curvearrowleft (\Sigma^n M, \Sigma^n N, \Sigma^n \alpha)$

Question: $\mathcal{I}^\alpha_M \circ \Sigma^n = ?$

Uninteresting answer: $\mathrm{Mod}^\omega_A \xrightarrow{\Sigma^n} \mathrm{Mod}^\omega_A$

$$(\mathrm{Mod}^\omega_A, \mathcal{I}^\alpha_M \circ \Sigma^n) \xrightarrow{\sim} (\mathrm{Mod}^\omega_A, \mathcal{I}^\alpha_M)$$

Notation: V a C_2-rep, X a spectrum w/ C_2-action

$\Sigma^V X = \Sigma^\infty S^V \otimes X$ w/ diagonal C_2-action.

$\sigma =$ sign rep of C_2

Lemma: X a spectrum w/ C_2-action

then $X^{tC_2} \xrightarrow{\sim} (\Sigma^\sigma X)^{tC_2}$

induced by $S^0 \longrightarrow S^\sigma$

pf. $C_{2+} \longrightarrow S^0 \longrightarrow S^\sigma$ cofibre sequence

$(C_{2+} \otimes X)^{tC_2} \longrightarrow X^{tC_2} \longrightarrow (\Sigma^\sigma X)^{tC_2}$ exact sequence.

free C_2-action. $\Rightarrow (C_{2+} \otimes X)^{tC_2} = 0$ $\square$

Lemma: X a spectrum w/ C_2-action

then $X^{tC_2} \xrightarrow{\sim} (\Sigma^\sigma X)^{tC_2}$

induced by $S^0 \longrightarrow S^\sigma$

Proposition: $(M, N, \alpha) \in \mathrm{Mod}_{NA}$, $n, m \in \mathbb{Z}$

then $\left(\text{Ϙ}_M^\alpha\right)^{[n+m]} \circ \Sigma^n \simeq \text{Ϙ}_{\Sigma^{m-n\sigma} M}^{\Sigma^m \alpha}$

$$\uparrow$$

$$\left(\Sigma^{m-n\sigma} M, \Sigma^m N, \Sigma^m N \xrightarrow{\Sigma^m \alpha} \Sigma^m M^{tC_2} \simeq (\Sigma^{m-n\sigma} M)^{tC_2}\right)$$

pf: $\left(\text{Ϙ}_M^\alpha\right)^{[m]} \simeq \text{Ϙ}_{\Sigma^m M}^{\Sigma^m \alpha} \rightsquigarrow$ suffices to show $m = 0$

In this case

$$\begin{tikzcd}
(\Sigma^n \circ \text{Ϙ}_M^\alpha \circ \Sigma^n)(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \Sigma^n \hom_A(\Sigma^n X, N) \simeq \hom_A(X, N) \arrow[d, "\alpha"] \\
\Sigma^n \text{Ϙ}_M^s(\Sigma^n X) \arrow[r] & \Sigma^n \hom_A(\Sigma^n X, M^{tC_2}) \simeq \hom_A(X, M^{tC_2})
\end{tikzcd}$$

Note: $\Sigma^n \hom_{A \otimes A}((\Sigma^n X) \otimes (\Sigma^n X), M)$

$$\simeq \Sigma^n \hom_{A \otimes A}(S^{n+n\sigma} \otimes X \otimes X, M)$$

$$\simeq \hom_{A \otimes A}(X \otimes X, \Sigma^{-n\sigma} M)$$

$\rightsquigarrow \Sigma^n \text{Ϙ}_M^s \circ \Sigma^n$ has the bilinear part

$\mathcal{B}_{\Sigma^{-n\sigma} M}$ $\square$

Proposition: $(M, N, \alpha) \in \mathrm{Mod}_{NA}$, $n, m \in \mathbb{Z}$

then $\left(\mathfrak{P}^{\alpha}_{M}\right)^{[n+m]} \circ \Sigma^{n} \simeq \mathfrak{P}^{\Sigma^{m}\alpha}_{\Sigma^{m-n\sigma}M}$

Corollary: $\Omega : \mathrm{Mod}^{\omega}_{A} \to \mathrm{Mod}^{\omega}_{A}$ refines to an equivalence

$$\left(\mathrm{Mod}^{\omega}_{A}, \left(\mathfrak{P}^{\alpha}_{M}\right)^{[2]}\right) \xrightarrow[\Omega]{\sim} \left(\mathrm{Mod}^{\omega}_{A}, \left(\mathfrak{P}^{\alpha}_{M}\right)^{[2]} \circ \Sigma\right)$$

$$\text{IS}$$

$$\left(\mathrm{Mod}^{\omega}_{A}, \mathfrak{P}^{\Sigma\alpha}_{\Sigma^{1-\sigma}M}\right)$$

Definition: k a E_∞ ring spectrum

a symmetry on k is a refinement of

$u : \mathbb{S} \to k$ to a C_2-equivariant map

$$\mathbb{S} \to \Sigma^{2-2\sigma} k.$$

we'll denote $\Sigma^{1-\sigma} k = -k$ "sign action".

$$(-k) \otimes_{k} (-k) \simeq k$$

examples: if $k = \mathbb{Z}$, $-k$ is the sign action.

example: if k is complex oriented, then it admits a symmetry.

nonexample: $k = \mathbb{S}$ does not admit a symmetry. why?

Corollary: k admits a symmetry, $A \in \mathrm{Alg}_{E_1}(\mathrm{Mod}_k)$, $M \in \mathrm{Mod}^{hC_2}_{A \otimes A}$

then $\Omega : \mathrm{Mod}^{\omega}_{A} \to \mathrm{Mod}^{\omega}_{A}$ refines to an

equivalence $\left(\mathrm{Mod}^{\omega}_{A}, \left(\mathfrak{P}^{s}_{M}\right)^{[2]}\right) \xrightarrow{\sim} \left(\mathrm{Mod}^{\omega}_{A}, \mathfrak{P}^{s}_{-M}\right)$

$$\left(\mathfrak{P}^{q}_{M}\right)^{[2]} \qquad \mathfrak{P}^{q}_{-M}$$

$$\left(\mathfrak{P}^{\geq m}_{M}\right)^{[2]} \qquad \left(\mathfrak{P}^{\geq m+1}_{-M}\right)$$

$1 \in \pi_0(MU)$ lift to $\pi_0((\Sigma^{2-2\sigma} MU)^{hC_2})$

HFPSS degenerates

$$H^{-p}(C_2, \pi_q(MU)) \Longrightarrow \pi_0((\Sigma^{2-2\sigma} MU)^{hC_2})$$

$C_2 \curvearrowright \pi_q(\Sigma^{2-2\sigma} MU)$ is trivial

$$S^{2\sigma} \to S^{2\sigma}$$