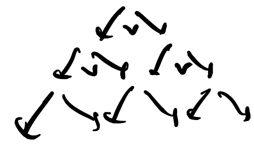
Q-construction, additivity theorem & isotropy separation

Recall: if $\mathcal{C}$ is stable ∞-cat, the Q-construction is a simplicial ∞-cat

$$Q(\mathcal{C}) : \Delta^{op} \longrightarrow \mathrm{Cat}_\infty^{ex}$$

$$Q_n(\mathcal{C}) = \mathrm{Fun}^{cart}(\mathrm{TwAr}(\Delta^n), \mathcal{C})$$


$\mathrm{TwAr}(\Delta^3)$

$$\mathrm{Span}(\mathcal{C}) = \mathrm{Core}\, Q(\mathcal{C}) \quad \text{complete Segal space.}$$

$$\underset{\substack{\uparrow \\ \text{K-theory space}}}{K(\mathcal{C})} = \Omega |\mathrm{Span}(\mathcal{C})|.$$

__Definition__ $(\mathcal{C}, \Omega)$ hermitian ∞-cat.

$$Q_n(\mathcal{C}, \Omega) = (Q_n\mathcal{C}, \Omega_n) \quad \text{where } \Omega_n = \Omega^{\mathrm{TwAr}(\Delta^n)}\big|_{Q_n\mathcal{C}}$$

explicitly: $\Omega_n(X \leftarrow Y \rightarrow Z \leftarrow \cdots) = \Omega(X) \underset{\Omega(Y)}{\times} \Omega(Z) \times \cdots$

Let $\mathcal{J}_n \subset \mathrm{TwAr}(\Delta^n)$: $\swarrow \searrow \swarrow \searrow \cdots$

So $Q_n(\mathcal{C}) = \mathcal{C}^{\mathcal{J}_n}$ $\qquad \mathcal{J}_n$ = poset of faces of $(\bullet\!-\!\bullet\!-\!\bullet \cdots \bullet\!-\!\bullet)$

Hence if $(\mathcal{C}, \Omega)$ is Poincaré, then $Q_n(\mathcal{C}, \Omega)$ is Poincaré.

Moreover: $D_{\Omega_n}(X \leftarrow Y \rightarrow Z \leftarrow \cdots) = D_\Omega(X) \leftarrow D_\Omega(X) \underset{D_\Omega(Y)}{\times} D_\Omega(Z) \rightarrow D_\Omega(Z) \leftarrow \cdots$

$\Rightarrow$ the simplicial maps $Q_n\mathcal{C} \to Q_m\mathcal{C}$ are duality-preserving:

e.g.:

$$\begin{array}{c} T \\ \swarrow \searrow \\ X \leftarrow Y \rightarrow Z \leftarrow W \rightarrow V \end{array} \quad \overset{D_{\Omega_2}}{\rightsquigarrow} \quad \begin{array}{c} DX \underset{DT}{\times} DV \\ \| \\ (DX \underset{DY}{\times} DZ) \underset{DZ}{\times} (DZ \underset{DW}{\times} DV) \\ \swarrow \qquad \searrow \\ D(X) \leftarrow \quad D(Z) \quad \rightarrow D(V) \end{array}$$

$$DZ = DY \underset{DT}{\times} DW$$

$$\begin{tikzcd}
\mathrm{Cat}_\infty^p \arrow[r, dashed, "Q_\cdot"] \arrow[d, hook] & (\mathrm{Cat}_\infty^p)^{\Delta^{op}} \arrow[d, hook] \\
\mathrm{Cat}_\infty^h \arrow[r, "Q_\cdot"] \arrow[d] & (\mathrm{Cat}_\infty^h)^{\Delta^{op}} \arrow[d] \\
\mathrm{Cat}_\infty^{ex} \arrow[r, "Q_\cdot"] & (\mathrm{Cat}_\infty^{ex})^{\Delta^{op}}
\end{tikzcd}$$

__Lemma:__ $Q(\mathcal{C}, \Omega)$ is a Segal object in Cat_∞^h.

From now on, $(\mathcal{C}, \Omega)$ Poincaré.

Definition. $\mathrm{Cob}(\mathcal{C},\Qoppa)$ = the ∞-category associated with the Segal space

$$\mathrm{Pn}\,\mathcal{Q}(\mathcal{C},\Qoppa^{[1]})$$

The Grothendieck–Witt space $\mathcal{GW}(\mathcal{C},\Qoppa) = \Omega|\mathrm{Cob}(\mathcal{C},\Qoppa)|$.

[Analogies:

Span	Cob
Core	Pn
K	GW

]

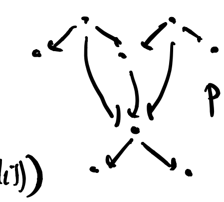
Lemma. Every face map in $\mathcal{Q}(\mathcal{C},\Qoppa)$ is a split Poincaré–Verdier projection.

Proof. split Verdier ✓

- outer face maps are restriction along $\mathcal{J}_{n-1} \xhookrightarrow{i} \mathcal{J}_n$, upward-closed inclusions
from last time $\Rightarrow$ $i^*: (\mathcal{C},\Qoppa)^{\mathcal{J}_n} \to (\mathcal{C},\Qoppa)^{\mathcal{J}_{n-1}}$ is split P–V.

- inner face maps are RKE along $\mathcal{J}_n \xrightarrow{p} \mathcal{J}_{n-1}$

$$(\mathcal{C},\Qoppa)^{\mathcal{J}_n} \xrightarrow{p_*} (\mathcal{C},\Qoppa)^{\mathcal{J}_{n-1}}$$



$$\Qoppa_{n-1}(\alpha: \mathcal{J}_{n-1}\to\mathcal{C}) = \lim_{i\in\mathcal{J}_{n-1}^{op}} \Qoppa(\alpha(i)) \underset{p \text{ cofinal}}{\uparrow =} \lim_{i\in\mathcal{J}_n^{op}} \Qoppa(\alpha(p(i))) \overset{=}{} \Qoppa_n(\alpha\circ p)$$

$$\begin{tikzcd} (\mathcal{C}^{\mathcal{J}_n})^{op} \arrow[d, shift right, "p_*"'] \arrow[dr, "\Qoppa_n"] & \\ (\mathcal{C}^{\mathcal{J}_{n-1}})^{op} \arrow[u, shift right, "p^*"'] \arrow[r, "\Qoppa_{n-1}"'] & \mathrm{Sp} \end{tikzcd}$$

$\Rightarrow$ $\Qoppa_{n-1}$ is LKE of $\Qoppa_n$ along p_*
($\equiv$ precomposition with right adjoint p^*). $\square$

Corollary. $\mathcal{F}: \mathrm{Cat}^{p}_{\infty} \to \mathcal{E}$ additive invariant. Then $\mathcal{F}(\mathcal{Q}(\mathcal{C},\Qoppa))$ is a Segal object in $\mathcal{E}$.

Pf.

$$\begin{tikzcd} \mathcal{Q}_{i+j}\mathcal{C} \arrow[r] \arrow[d] & \mathcal{Q}_i\mathcal{C} \arrow[d, "\text{outer face map}"] \\ \mathcal{Q}_j\mathcal{C} \arrow[r] & \mathcal{Q}_0\mathcal{C} \end{tikzcd} \quad \overset{\mathcal{F}}{\rightsquigarrow} \quad \text{cartesian square.} \qquad \square.$$

Def $\mathcal{F}: \mathrm{Cat}^{p}_{\infty} \to \mathcal{S}$ additive.

$\mathrm{Cob}^{\mathcal{F}}: \mathrm{Cat}^{p}_{\infty} \to \mathrm{Cat}_\infty$ $\quad (\mathcal{C},\Qoppa) \mapsto$ ∞-cat associated with the Segal space $\mathcal{F}(\mathcal{Q}(\mathcal{C},\Qoppa^{[1]}))$.

Examples:
- $\mathrm{Cob}^{\mathrm{Pn}} = \mathrm{Cob}$
- $\mathrm{Cob}^{\mathrm{Core}}(\mathcal{C},\Qoppa) = \mathrm{Span}(\mathcal{C})$
- $\mathrm{Cob}(\mathrm{Hyp}(\mathcal{C})) = \mathrm{Span}(\mathcal{C})$ because $\mathrm{Pn}\circ\mathrm{Hyp} = \mathrm{Core}$.

• $\mathrm{Cob}(\mathcal{C}, \mathfrak{P}^s_B) = \mathrm{Span}(\mathcal{C})^{hC_2}$ because $\mathrm{Pn}(\mathcal{C}, \mathfrak{P}^s_B) = \mathrm{Core}(\mathcal{C})^{hC_2}$.

via $D_{\mathfrak{P}}$

Since $F : \mathrm{Cat}^p_\infty \to \mathrm{Mon}_{E_\infty}(\mathcal{S})$, so $\mathrm{Cob}^F : \mathrm{Cat}^p_\infty \to \mathrm{Mon}_{E_\infty}(\mathrm{Cat}_\infty)$

$$|\mathrm{Cob}^F| = |F \circ Q| : \mathrm{Cat}^p_\infty \to \mathrm{Mon}_{E_\infty}(\mathcal{S})$$

<u>Prop.</u> $|\mathrm{Cob}^F(\mathcal{C}, \mathfrak{P})|$ is a group with $-[x] = [(\mathrm{id}_{\mathcal{C}}, -\mathrm{id}_{\mathfrak{P}})_*(x)]$.

<u>Additivity Theorem.</u>

If $F : \mathrm{Cat}^p_\infty \to \mathcal{S}$ is additive, then $|\mathrm{Cob}^F| : \mathrm{Cat}^p_\infty \to \mathcal{S}$ is additive.

<u>Proof strategy</u>: $|\mathrm{Cob}^F| = |F \circ Q|$, $F \circ Q : \mathrm{Cat}^p_\infty \to \mathcal{S}^{\Delta^{op}}$ is additive (because Q_n preserves split P-V projections)

so we need to commute $|\cdot|$ with some pullbacks

<u>Theorem</u>: Let $(p, \eta) : (\mathcal{C}, \mathfrak{P}) \to (\mathcal{C}', \mathfrak{P}')$ is split P-V projection.

Then $(p,\eta)_* : FQ(\mathcal{C}, \mathfrak{P}) \to FQ(\mathcal{C}', \mathfrak{P}')$ is a realization-fibration of simplicial spaces

$$\begin{tikzcd} Y'_\bullet \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y_\bullet \arrow[d, "p"] \\ X'_\bullet \arrow[r, "\forall f"'] & X_\bullet \end{tikzcd}$$

<u>Lemma</u> (Steimle 2018) bicartesian fibrations of Segal spaces are realization-fibrations.

Since $Q \simeq Q \circ op$, it suffices to show that $(p, \eta)_*$ is a cocartesian fibration.

$\mathcal{C} \xrightarrow{p} \mathcal{C}'$ split Verdier $\Rightarrow$ p bicartesian fibration

$$\begin{tikzcd} \mathcal{C} \arrow[d, "p"] & X & Y \arrow[l, "p\text{-cart}"'] \arrow[r, "p\text{-cocart}"] & Z \\ \mathcal{C}' & X' & Y' \arrow[l] \arrow[r] & Z' \end{tikzcd}$$

[$X_\bullet \xrightarrow{p} Y_\bullet$ Segal space $\qquad X_1^{cocart} \subset X_1$ p-cocart edges

p cocartesian $\iff$ $X_1^{cocart} \to Y_1 \times_{Y_0} X_0$ is surjective on π_0. ($s = d_1$)]

Idea of proof: let $\mathcal{E} \subset Q_1(\mathcal{C})$ full subcategory on $X \xleftarrow{p\text{-cart}} Y \xrightarrow{p\text{-cocart}} Z$

We have split P-V squares:

1) $$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "p"] \\ Q_1\mathcal{C}' \arrow[r, "d_1"] & \mathcal{C}' \end{tikzcd}$$

2) $$\begin{tikzcd} \mathcal{E} \times_{Q_1\mathcal{C}, d_2} Q_2\mathcal{C} \arrow[r, "(\mathrm{id}, d_1)"] \arrow[d] & \mathcal{E} \times_{Q_1\mathcal{C}, d_1} Q_1\mathcal{C} \arrow[d] \\ Q_2\mathcal{C}' \arrow[r, "(d_2, d_1)"] & Q_1\mathcal{C}' \times_{d_1, \mathcal{C}', d_1} Q_1\mathcal{C}' \end{tikzcd}$$ ← B.C of $Q_1(p)$.

Apply $\mathcal{F}(-)$:

2) implies that $\mathcal{F}(\mathcal{E}, \Omega_1) \longrightarrow \mathcal{F}(Q_1(\mathcal{C}, \Omega))$

$\cup$

$\dashrightarrow \mathcal{F}(Q_1(\mathcal{C}, \Omega))^{\text{cocart}}$

1) implies that $\mathcal{F}(\mathcal{E}, \Omega_1) \xrightarrow{\simeq} \mathcal{F}(Q_1(\mathcal{C}, \Omega)) \times_{\mathcal{F}(\mathcal{C}', \Omega')} \mathcal{F}(\mathcal{C}, \Omega)$. ☐

<u>Cobordism of Poincaré functors</u>. $\mathcal{C} \longrightarrow Q_1\mathcal{D}$ $\quad 0 \to \mathrm{fib}(h \to f) \to h$, $f \leftarrow h \rightarrow g$ filtration of g.

associated graded: $\mathrm{fib}(h \to f)$, f, $\mathrm{cofib}(h \to g)$

$\Rightarrow \quad g_* \simeq f_* + \mathrm{fib}(h \to f)_* + \mathrm{cofib}(h \to g)_*: K(\mathcal{C}) \longrightarrow K(\mathcal{D})$.

$g_* - f_* \simeq \mathrm{fib}(h \to f)_* - \mathrm{fib}(h \to g)_*$

<u>Theorem</u> If $f \leftarrow h \rightarrow g$ is a functor $(\mathcal{C}, \Omega) \longrightarrow Q_1(\mathcal{D}, \Phi)$ and $\mathcal{F}: \mathrm{Cat}^{p}_{\infty} \to \mathcal{S}$ is a <u>group-like</u> additive functor, then

$$g_* - f_* \simeq \big(\mathrm{fib}(h \to f) + \mathrm{cofib}(h \to g)\big)_*: \mathcal{F}(\mathcal{C}, \Omega) \longrightarrow \mathcal{F}(\mathcal{D}, \Phi).$$

<u>Pf sketch</u>: universal case: $\mathrm{id}: Q_1\mathcal{C} \longrightarrow Q_1\mathcal{C}$ $\quad d_1 \leftarrow \varphi \rightarrow d_0 : Q_1\mathcal{C} \to \mathcal{C}$.

$i: \mathcal{C} \hookrightarrow Q_1\mathcal{C} \quad \dashv \quad p: Q_1\mathcal{C} \to \mathcal{C}$

$x \longmapsto (0 \leftarrow x = x) \qquad (x \leftarrow w \rightarrow y) \longmapsto \mathrm{fib}(w \to x)$

Hyp: $\mathrm{Cat}^{\mathrm{ex}}_{\infty} \to \mathrm{Cat}^{p}_{\infty}$ both left and right adjoint to the forgetful functor,

$i_{\mathrm{hyp}}: \mathrm{Hyp}(\mathcal{C}) \to Q_1(\mathcal{C}, \Omega) \qquad p^{\mathrm{hyp}}: Q_1(\mathcal{C}, \Omega) \to \mathrm{Hyp}(\mathcal{C})$.

split P-V. sequence: $$\begin{tikzcd} \mathcal{C} \arrow[r, hook, "s_0"] & Q_1\mathcal{C} \arrow[l, bend left, "d_0 \circ d_1"] \arrow[r, "p^{\mathrm{hyp}}"] & \mathrm{Hyp}(\mathcal{C}) \arrow[l, bend left, "i_{\mathrm{hyp}}"] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{F}(\mathcal{C}) \oplus \mathcal{F}(\mathrm{Hyp}\,\mathcal{C}) \arrow[r, "\sim", "(s_0,\, i_{hyp})"'] \arrow[rr, bend right, "\mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}"'] & \mathcal{F}(Q_1\mathcal{C}) \arrow[r, "\sim", "(d_0,\, p^{hyp})"'] & \mathcal{F}(\mathcal{C}) \oplus \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \end{tikzcd}.$$

$\Rightarrow \quad \mathrm{id}_{\mathcal{F}(Q_1\mathcal{C})} \simeq s_0 d_0 + i_{hyp} p^{hyp}$

apply $d_1 \circ (-)$: $\quad d_1 \simeq d_0 + \underbrace{d_1 i_{hyp} p^{hyp}}_{\mathrm{fib}(\varphi \to d_1) \oplus \mathrm{cofib}(\varphi \to d_0)}$ $\square$

Def. • An isotropic subcategory $\mathcal{L} \subset \mathcal{C}$ is a full stable subcategory s.t.:

1) $Ϙ|_{\mathcal{L}} = 0 \quad (\Rightarrow \mathcal{L} \subset \mathcal{L}^\perp)$

2) $\mathcal{L}^{op} \hookrightarrow \mathcal{C}^{op} \xrightarrow{D} \mathcal{C} \to \mathcal{C}/\mathcal{L}^\perp$ is an equivalence

where $\mathcal{L}^\perp = \{x \in \mathcal{C} \mid B_Ϙ(x,y) = 0 \ \forall y \in \mathcal{L}\}$.

• The homology category $\mathrm{Hlgy}(\mathcal{L})$ is the cofiber of $(\mathcal{L}, Ϙ) \hookrightarrow (\mathcal{L}^\perp, Ϙ)$ in Cat_∞^h.

• $\mathcal{L}$ is Lagrangian if $\mathrm{Hlgy}(\mathcal{L}) = 0$ i.e. $\mathcal{L} = \mathcal{L}^\perp$.

• $(\mathcal{C}, Ϙ)$ is metabolic if it has a Lagrangian

Properties: $\mathcal{L} \overset{i}{\hookrightarrow} \mathcal{C}$ isotropic.

• i has a right adjoint p with $\ker(p) = D(\mathcal{L}^\perp)$.

• $(\mathcal{L}^\perp)^\perp = \mathcal{L}$

• $\mathcal{L}^\perp \cap D(\mathcal{L}^\perp) \hookrightarrow \mathcal{L}^\perp \to \mathcal{L}^\perp/\mathcal{L}$ refines to an equivalence

$(\mathcal{L}^\perp \cap D(\mathcal{L}^\perp), Ϙ) \simeq \mathrm{Hlgy}(\mathcal{L})$ in Cat_∞^p.

Hence $\iota : \mathrm{Hlgy}(\mathcal{L}) \hookrightarrow (\mathcal{C}, Ϙ)$

Ex: • $\mathcal{C} \times 0 \subset \mathrm{Hyp}(\mathcal{C})$ is Lagrangian

• $\mathcal{C} \overset{s_0}{\hookrightarrow} \mathrm{Met}(\mathcal{C}, Ϙ)$ $\longmapsto$ "

• $i : \mathcal{C} \hookrightarrow Q_1\mathcal{C}$ is isotropic $\quad \mathcal{C}^\perp = \{ y \leftarrow x = x \}$ $\quad \mathrm{Hlgy}(\mathcal{C} \subset Q_1\mathcal{C}) = \mathcal{C}$

$x \mapsto 0 \leftarrow x = x$ $\quad D_{Ϙ_1}(\mathcal{C}^\perp) = \{ y = y \to x \}$ $\quad s_0 \downarrow$ $Q_1\mathcal{C}$

Theorem (Isotropy separation).

Let $\mathcal{L} \overset{i}{\subset} \mathcal{C}$ isotropic subcategory. $\mathcal{F} : \mathrm{Cat}_\infty^p \longrightarrow \mathcal{E}$ grouplike additive.

Then

$$\mathcal{F}(\mathcal{C},\varphi) \xleftarrow[(i_{hyp},\,\iota)]{\simeq} \mathcal{F}(\mathrm{Hyp}(\mathcal{L})) \oplus \mathcal{F}(\mathrm{Hlgy}(\mathcal{L}))$$

In particular if $\mathcal{L}$ Lagrangian, $\mathcal{F}(\mathcal{C},\varphi) = \mathcal{F}(\mathrm{Hyp}(\mathcal{L}))$.

Corollary. $\mathcal{F} : \mathrm{Cat}_\infty^p \longrightarrow \mathcal{E}$ grouplike additive. Then

$$\mathcal{F}(\mathcal{Q}_\bullet(\mathcal{C},\varphi)) \simeq \mathcal{F}(\mathrm{Hyp}(\mathcal{C}))^\bullet \circledast \mathcal{F}(\mathcal{C},\varphi)$$

action of $\mathrm{Hyp}(\mathcal{C}) \longrightarrow (\mathcal{C},\varphi)$