Structure theory for additive functors

Recall: $\mathcal{Q}(\mathcal{C},\Phi)\colon \quad [n] \longmapsto \mathcal{Q}_n(\mathcal{C},\Phi) := \big(\overset{\mathcal{Q}_n\mathcal{C}}{\overset{\shortparallel}{\mathrm{Fun}^{\mathrm{cart}}(\mathrm{Tw}[n],\mathcal{C})}}\,,\ \Phi^{\mathrm{Tw}[n]}|_{\dots}\big)$ Segal object in Cat_∞^p

$\mathcal{F}\colon \mathrm{Cat}_\infty^p \to \mathcal{S}$ additive $\rightsquigarrow$ $\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Phi) := \infty$-cat. associated to $\mathcal{F}\mathcal{Q}(\mathcal{C},\Phi^{[1]})$.

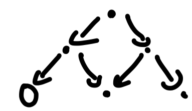
Additivity thm.: $|\mathrm{Cob}^{\mathcal{F}}|$ additive

Reminder:

$$\begin{tikzcd}
& \mathcal{S} \arrow[dl, shift right, "\mathrm{const}"'] \arrow[dr, shift left] & \\
s\mathcal{S} \arrow[ur, shift right, "\mathrm{colim}"'] \arrow[ur, phantom, "\perp"] \arrow[rr, shift right, "\infty\text{-cat}(-)"'] \arrow[rr, phantom, "\top"] & & \mathrm{Cat}_\infty \arrow[ul, shift left, "|-|"] \arrow[ul, phantom, "\top"] \arrow[ll, shift right, "N"']
\end{tikzcd}$$

§1 Universal properties of $|\mathrm{Cob}^-|$

$$\mathrm{Null}(\mathcal{C},\Phi^{[1]}) := \mathrm{fib}_0\big(\mathrm{dec}\,\mathcal{Q}(\mathcal{C},\Phi^{[1]}) := \mathcal{Q}(\mathcal{C},\Phi^{[1]})\circ([0]*-) \longrightarrow (\mathcal{C},\Phi^{[1]})\big)$$



This augmented simplicial object is split (= admits an additional degeneracy).
Hence, so is $\mathrm{Null}(\mathcal{C},\Phi^{[-1]}) \longrightarrow 0$.

$$\begin{tikzcd}
(\mathcal{C},\Phi) \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)"] & \mathrm{Null}(\mathcal{C},\Phi^{[1]}) \arrow[d, "\pi"] \arrow[dr] & \\
0 \arrow[r] & \mathcal{Q}(\mathcal{C},\Phi^{[1]}) & \mathrm{dec}\,\mathcal{Q}(\mathcal{C},\Phi^{[1]}) \arrow[l, "d_0"]
\end{tikzcd}$$

$\ker\pi$:

$$\begin{tikzcd}
& & X \arrow[dl, equal] \arrow[dr] & & \\
& X \arrow[dl] \arrow[dr] & & 0 \arrow[dl] \arrow[dr] & \\
0 & & 0 & & 0
\end{tikzcd}$$

$\Rightarrow$ $(*)$ degreewise split PV.

Fact: $X_\bullet \to X_{-1}$ split $\Rightarrow$ $\mathrm{colim}\, X \simeq X_{-1}$. $\rightsquigarrow$ Obtain $\mathcal{F}(\mathcal{C},\Phi) \xrightarrow{\beta} \Omega|\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Phi)|$.

Definition

$$\left.\begin{aligned}\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Phi) := \{\, \mathcal{F}(\mathcal{C},\Phi),\ |\mathrm{Cob}^{\mathcal{F}}(\mathcal{C},\Phi)|,\ |\mathrm{Cob}^{|\mathrm{Cob}^{\mathcal{F}}|}(\mathcal{C},\Phi)|,\ \dots \,\} \\ \text{bonding maps given by } \beta \end{aligned}\right\} \in \mathrm{PSp}$$

May interpret Cob^- as functor

$$\begin{tikzcd}
& \mathrm{Fun}(\mathrm{Cat}_\infty^p, \mathrm{PSp}) \\
\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^p, \mathcal{S}) \arrow[ur] \arrow[dr] & \\
& \mathrm{PSp}(\mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}_\infty^p, \mathcal{S}))
\end{tikzcd}$$

Theorem

There are adjunctions

$$\mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathcal{S}) \simeq \mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathrm{Mon}_{E_\infty}) \underset{\Omega|\mathrm{Cob}^-| =: (-)^{grp}}{\overset{\perp}{\rightleftarrows}} \mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathrm{Grp}_{E_\infty}) \underset{\mathrm{Cob}^-}{\overset{\Omega^\infty}{\overset{\perp}{\rightleftarrows}}} \mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathrm{Sp})$$

$$\cong \qquad\qquad\qquad\qquad \cong$$

$$\mathrm{Grp}_{E_\infty}(\mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathcal{S})) \underset{\Sigma^\infty}{\overset{\Omega^\infty}{\overset{\perp}{\rightleftarrows}}} \mathrm{Sp}(\mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathcal{S}))$$

Fact: $\mathcal{A}$ semi-additive s.th. Ω, Σ exist and $\Sigma: \mathcal{A}_{grp} \to \mathcal{A}_{grp}$ is fully faithful.

$\Rightarrow$ $id \to \Omega\Sigma$ is the unit of an adjunction $\mathcal{A} \underset{\Omega\Sigma}{\overset{\perp}{\rightleftarrows}} \mathcal{A}_{grp}$

Idea: $\Omega A, \Sigma A \in \mathcal{A}_{grp}$ eg: $\Sigma A \oplus \Sigma A \xrightarrow{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}} \Sigma A \oplus \Sigma A$ equiv? $\rightsquigarrow$ $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$ exists!

$$\begin{array}{ccc} \mathrm{Hom}(G, \Omega\Sigma A) & \xrightarrow{\Omega\Sigma u} & \mathrm{Hom}(G, \Omega\Sigma\Omega\Sigma A) \\ \simeq \downarrow \Sigma & \swarrow & \\ \mathrm{Hom}(\Sigma G, \Sigma\Omega\Sigma A) & & \end{array}$$

$$\Omega\Sigma A \underset{u_{\Omega\Sigma}}{\overset{\Omega\Sigma u}{\rightrightarrows}} \Omega\Sigma\Omega\Sigma A$$

both split by Ωc_Σ.

Suffices to show:

Theorem

$\mathcal{F}: \mathrm{Cat}^p_\infty \to \mathcal{S}$ additive

Then

$$\begin{array}{ccc} \mathcal{F}(-) & \longrightarrow & \mathrm{colim}\, \mathcal{F}\mathrm{Null}(-^{[1]}) \simeq * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{colim}\, \mathcal{F}Q(-^{[1]}) \simeq |\mathrm{Cob}^{\mathcal{F}}| \end{array}$$

i) is a pushout in $\mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathcal{S})$ [ie $|\mathrm{Cob}^{\mathcal{F}}| \simeq \Sigma\mathcal{F}$]

ii) is a pullback in $\mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathcal{S})$ if $\mathcal{F}$ is group-like

[ie $\mathcal{F} \xrightarrow{\sim} \Omega|\mathrm{Cob}^{\mathcal{F}}| \overset{i)}{\simeq} \Omega\Sigma\mathcal{F}$

$\Rightarrow \underbrace{\mathrm{Hom}(G, \mathcal{F}) \xrightarrow{\Sigma} \mathrm{Hom}(\Sigma G, \Sigma\mathcal{F})}_{\sim} \simeq \mathrm{Hom}(G, \Omega\Sigma\mathcal{F})$]

Proof

i) Show that
$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "(*)_n"] & \mathcal{F}\mathrm{Null}_n(-^{[1]}) \arrow[d] \\ * \arrow[r] & \mathcal{F}Q_n(-^{[1]}) \end{tikzcd}$$
is a pushout for all n.

$\swarrow \searrow \swarrow \ldots \swarrow \searrow$

$$(-)_{y_n} =: dQ_n : \mathrm{Cat}^p_\infty \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathrm{Cat}^p_\infty : Q_n \simeq (-)^{y_n} \quad \Rightarrow \quad - \circ Q_n : \mathrm{Fun}(\mathrm{Cat}^p_\infty, \mathcal{S}) \longrightarrow \mathrm{Fun}(\mathrm{Cat}^p_\infty, \mathcal{S})$$
is a left adjoint, so preserves colimits

assemble to a cosimplicial object

Null_n also has left adjoint $d\mathrm{Null} := \mathrm{cofib}(dQ_0 \longrightarrow dQ_{n+1})$

$\Rightarrow$ $(*)_n$ is preserves colimits as a functor on $\mathrm{Fun}(\mathrm{Cat}^p_\infty, \mathcal{S})$.

So need only check $\mathcal{F} = j(\mathcal{C}, \varphi)$, $\quad j : (\mathrm{Cat}^p_\infty)^{op} \longrightarrow \mathrm{Fun}^{add}(\mathrm{Cat}^p_\infty, \mathcal{S})$ Yoneda

ie consider
$$\begin{tikzcd} j(\mathcal{C}, \varphi) \arrow[r] \arrow[d] & j(d\mathrm{Null}_n(\mathcal{C}, \varphi^{[-1]})) \arrow[d] \\ * \arrow[r] & j(dQ_n(\mathcal{C}, \varphi^{[-1]})) \end{tikzcd}$$
$$j(dQ_n(\mathcal{C}, \varphi^{[-1]})) \simeq \mathrm{Hom}(dQ_n(\mathcal{C}, \varphi), -^{[1]}) \simeq \mathrm{Hom}((\mathcal{C}, \varphi), Q_n(-^{[1]})) = j(\mathcal{C}, \varphi) \circ Q_n(-^{[1]})$$

$$dQ_n(\mathcal{C}, \varphi^{[-1]}) \longrightarrow d\mathrm{Null}_n(\mathcal{C}, \varphi^{[-1]}) \longrightarrow (\mathcal{C}, \varphi)$$
left adjoint of a split PV seq., so cofibre seq.

This is even split PV. $\Longrightarrow$ i)

ii) $\mathcal{F}$ additive $\Rightarrow$ $(*)_n$ is a pullback.

$$\begin{tikzcd} \mathcal{F}\mathrm{Null}_n(-^{[1]}) \arrow[r] \arrow[d, "\pi_n"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{F}\mathrm{Null}_m(-^{[1]}) \arrow[d, "\pi_m"] \\ \mathcal{F}Q_n(-^{[1]}) \arrow[r] & \mathcal{F}Q_m(-^{[1]}) \end{tikzcd}$$

Show that $[n] \mapsto (*)_n$ is equifibred for all n, ie $\forall\, [m] \longrightarrow [n] \in \Delta$

wlog: consider only face maps.

Under the identifications of the previous talk, obtain

$$\begin{tikzcd} \mathcal{F}\mathrm{Hyp}(\mathcal{C})^{n+1} \arrow[r] \arrow[d] & \mathcal{F}\mathrm{Hyp}(\mathcal{C})^{n} \arrow[d] \\ \mathcal{F}\mathrm{Hyp}(\mathcal{C})^{n} \oplus \mathcal{F}(\mathcal{C}, \varphi^{[1]}) \arrow[r] & \mathcal{F}\mathrm{Hyp}(\mathcal{C})^{n-1} \oplus \mathcal{F}(\mathcal{C}, \varphi^{[1]}) \end{tikzcd}$$

vertical maps: omit 0-th factor

horizontal maps: bar construction

Rezk's equifibration criterion $\Rightarrow$ ii).

$\square$

Think $\mathcal{F} = \mathrm{Pn}$ & $\mathrm{Hyp}\,\mathscr{C}$

Proposition

$\mathcal{F}$ group-like, additive, $(\mathscr{C},\varphi)$ admits Lagrangian $\Longrightarrow$ $\mathrm{Cob}^{\mathcal{F}}(\mathscr{C},\varphi)$ connective.

Proof

$n \geq 1$: $\quad \pi_{-n}\mathrm{Cob}^{\mathcal{F}}(\mathscr{C},\varphi) \simeq \pi_0 |\mathrm{Cob}^{|\mathrm{Cob}^{\cdot^{\cdot^{\cdot |\mathrm{Cob}^{\mathcal{F}}|}\cdot}\cdot}}|(\mathscr{C},\varphi) \overset{\substack{\text{isotropy}\\ \text{sep.}}}{\simeq} \pi_0 | \dots |(\mathrm{Hyp}(\mathcal{L}))$

Observe a) $\quad \mathrm{coker}\left(\pi_0\, G\,\mathrm{Met}(\mathcal{D},\phi^{[1]}) \xrightarrow{\mathrm{met}} \pi_0\, G(\mathcal{D},\phi^{[1]})\right) \twoheadrightarrow \pi_0|\mathrm{Cob}^G(\mathcal{D},\phi)|$:

$$\simeq$$

$$\begin{tikzcd}
\mathrm{coeq}\Big(\pi_0 G Q_1(\mathcal{D},\phi^{[1]}) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \pi_0 G(\mathcal{D},\phi^{[1]}) \Big) \\
\mathrm{Met}(\mathcal{D},\phi^{[1]}) \arrow[u] \arrow[ur, shift left, "\mathrm{met}"] \arrow[ur, shift right, "0"'] &
\end{tikzcd}$$

$(X \to Y) \mapsto (0 \leftarrow X \rightarrow Y)$

b) $\mathrm{Met}(\mathrm{Hyp}(\mathcal{D})) \simeq \mathrm{Hyp}(\mathrm{Ar}\,\mathcal{D})$ by direct computation $\square$

§2 Bordism invariant functors

Recall: bordism

$$\begin{tikzcd}
 & (\mathcal{C}',Ϙ') \\
(\mathcal{C},Ϙ) \arrow[ur, "(f,\eta)"] \arrow[r] \arrow[dr, "(g,\xi)"'] & Q_1(\mathcal{C}',Ϙ') \arrow[u, "d_1"'] \arrow[d, "d_0"] \\
 & (\mathcal{C}',Ϙ')
\end{tikzcd}$$

(f,η) bordism equivalence if it admits an inverse up to bordism

$\mathrm{Fun}^{\mathrm{bord}}(\mathrm{Cat}^p_\infty, \mathrm{Sp}) \subseteq \mathrm{Fun}^{\mathrm{add}}(\mathrm{Cat}^p_\infty, \mathrm{Sp})$ full subcat of bordism invariant functors

Lemma

$\mathcal{F}$ group-like additive. Tfae:

i) $\mathcal{F}$ bordism invariant

ii) $\mathcal{F}(\mathcal{C},Ϙ) \xrightarrow{\mathcal{F}(s)} \mathcal{F}(Q_1(\mathcal{C},Ϙ))$ equiv. $\forall (\mathcal{C},Ϙ)$

iii) $\mathcal{F}(\mathrm{Met}(\mathcal{C},Ϙ)) \simeq 0 \quad \forall (\mathcal{C},Ϙ)$

iv) $\mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \simeq 0 \quad \forall \mathcal{C}$

Proof

i) $\Longleftrightarrow$ ii) s bordism equiv.:

$$Q_1\mathcal{C} \longrightarrow Q_1 Q_1 \mathcal{C}$$

$$X \leftarrow Y \rightarrow Z \quad \longmapsto \quad \begin{tikzcd}
X & Y \arrow[l] \arrow[r] & Z \\
X \arrow[u] \arrow[d] & Y \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \arrow[u] \arrow[d] \\
X & X \arrow[l] \arrow[r] & X
\end{tikzcd} \quad \xmapsto{Ϙ} \lim \quad \begin{tikzcd}
ϘX \times_{ϘY} ϘZ \arrow[d] \\
ϘX \\
ϘX \arrow[u, "\sim"']
\end{tikzcd}$$

Conversely, d_0, d_1 are retractions to s.

i) $\Longrightarrow$ iii)

$$\mathrm{Met}(\mathcal{C},Ϙ) \longrightarrow Q_1 \mathrm{Met}(\mathcal{C},Ϙ)$$

$$X \rightarrow Y \quad \longmapsto \quad \begin{tikzcd}
X \arrow[r] & Y \\
X \arrow[r] \arrow[u] \arrow[d] & X \arrow[u] \arrow[d] \\
0 \arrow[r] & 0
\end{tikzcd} \qquad \Longrightarrow \mathrm{Met}(\mathcal{C},Ϙ) \simeq_b 0$$

iii) $\Longleftrightarrow$ iv) by isotropy separation

iv) $\Longrightarrow$ ii) $\quad \mathcal{F}Q_1(\mathcal{C},Ϙ) \xleftarrow[(s,\ldots)]{\simeq} \mathcal{F}(\mathcal{C},Ϙ) \oplus \mathcal{F}\mathrm{Hyp}(\mathcal{C})$

$\square$

__Fact:__ $(\mathcal{C}, Ϙ) \in \mathrm{Cat}^{p}_{\infty} \Rightarrow \quad \mathrm{Hyp}(\mathcal{C}) = \mathcal{C} \oplus \mathcal{C}^{op} \xrightarrow{D^{op} \oplus D} \mathcal{C}^{op} \oplus \mathcal{C} \overset{flip}{\simeq} \mathcal{C} \oplus \mathcal{C}^{op} = \mathrm{Hyp}(\mathcal{C})$

refines to a C_2-object $\overline{\mathrm{Hyp}}$ in $\mathrm{Cat}^{p}_{\infty}$.

The transformations $\mathrm{Hyp}(\mathcal{C}) \longrightarrow (\mathcal{C}, Ϙ) \longrightarrow \mathrm{Hyp}(\mathcal{C})$ refine to equivariant maps wrt the trivial action on $(\mathcal{C}, Ϙ)$.

__Definition__ $\quad \mathcal{F}^{hyp} := \mathcal{F} \circ \overline{\mathrm{Hyp}} : \mathrm{Cat}^{p}_{\infty} \longrightarrow \mathrm{Fun}(BC_2, \mathrm{Sp})$.

__Theorem__

There is an adjunction $\mathrm{Fun}^{add}(\mathrm{Cat}^{p}_{\infty}, \mathrm{Sp}) \underset{(-)^{bord}}{\overset{\top}{\rightleftarrows}} \mathrm{Fun}^{bord}(\mathrm{Cat}^{p}_{\infty}, \mathrm{Sp})$ such that

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^{bord} \arrow[d] \\ (\mathcal{F}^{hyp})^{hC_2} \arrow[r] & (\mathcal{F}^{hyp})^{tC_2} \end{tikzcd}$$

is bicartesian.

__Proof__

$\mathcal{F}^{bord} := \mathrm{cof}(\mathcal{F}^{hyp}_{hC_2} \longrightarrow \mathcal{F})$.

Bordism invariance:

$$\begin{tikzcd} \mathcal{F}^{hyp}_{hC_2}(\mathrm{Hyp}(\mathcal{C})) \arrow[r] & \mathcal{F}(\mathrm{Hyp}(\mathcal{C})) \\ \mathcal{F}(\mathrm{Hyp}(\mathcal{C}) \otimes C_2)_{hC_2} \arrow[u, phantom, "\simeq" rotate=90] \arrow[ur, "\sim"'] & \end{tikzcd}$$

Adjointness: $\quad G$ bordism invariant $\overset{!}{\Rightarrow} \mathrm{Nat}(\mathcal{F}^{hyp}_{hC_2}, G) \simeq 0$

__Fact:__ $\mathrm{Hyp} \dashv \mathrm{fgt} \dashv \mathrm{Hyp} \qquad$ (so $\mathrm{Hyp} \circ \mathrm{fgt} \dashv \mathrm{Hyp} \circ \mathrm{fgt}$)

$\Rightarrow \mathrm{Nat}(\mathcal{F}^{hyp}, G) \simeq \mathrm{Nat}(\mathcal{F}, G^{hyp}) \overset{\text{Lemma}}{\simeq} \mathrm{Nat}(\mathcal{F}, 0) \simeq 0$

$\Rightarrow \mathrm{Nat}(\mathcal{F}^{hyp}_{hC_2}, G) \simeq \mathrm{Nat}(\mathcal{F}^{hyp}, G)^{hC_2} \simeq 0$. $\quad \square$

Definition

$$\varrho : \quad [n] \longmapsto \varrho_n(\mathcal{C},\Phi) := (\mathcal{C},\Phi)^{P_{\neq\emptyset}[n]^{op}} \quad \in s\mathrm{Cat}^{p}_{\infty}$$

$$\mathrm{ad}\,\mathcal{F} := |\mathcal{F}\circ\varrho|$$

Proposition

$b: \mathcal{F} \to \mathrm{ad}\,\mathcal{F}$ exhibits ad as bordification.

Proof

Show:

a) ad preserves colimits

b) $\mathcal{F}$ bordism invariant $\Rightarrow$ b equiv.

c) $\mathrm{ad}\,\mathcal{F}^{hyp} \simeq 0$

$\Rightarrow$

$$\begin{tikzcd}
\mathcal{F}^{hyp}_{hC_2} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}^{bord} \arrow[d, "\simeq", "b)"'] \\
\mathrm{ad}(\mathcal{F}^{hyp}_{hC_2}) \arrow[r] \arrow[d, phantom, "\wr\ a)"] & \mathrm{ad}\,\mathcal{F} \arrow[r] & \mathrm{ad}\,\mathcal{F}^{bord} \\
(\mathrm{ad}\,\mathcal{F}^{hyp})_{hC_2} \arrow[d, phantom, "\wr\ c)"] & & \\
0 & &
\end{tikzcd}$$

cofibre seq. by a)

ad b)

$$\varrho_n(\mathcal{C},\Phi) \xrightarrow{\text{extend by } 0} \Big(\mathrm{Fun}(P[n]^{op},\mathcal{C}),\ \text{total fib of } \Phi^{[1]}\Big) \xrightarrow{ev_\emptyset} (\mathcal{C},\Phi^{[1]}) \quad \text{split PV}$$

$$\wr$$

$$\mathrm{Met}^{(n+1)}(\mathcal{C},\Phi^{[1]}) \overset{\mathcal{F}}{\longmapsto} 0 \quad \Rightarrow \quad \mathcal{F}_\varrho(\mathcal{C},\Phi) \text{ constant}$$

ad c)

$$\begin{tikzcd}
\mathcal{F}\mathrm{Hyp}(\mathcal{C}^{P[n]^{op}}) \arrow[r, phantom, "\overset{\text{isotropy seq.}}{\simeq}"] \arrow[drr, "ev_\emptyset"] & \mathcal{F}(\mathrm{Met}(\mathcal{C}^{P[n]^{op}},\dots)) \simeq & \mathcal{F}(\mathrm{Met}^{(n+2)}(\mathcal{C},\Phi^{[1]})) \arrow[d, "\text{split augm. simpl. space}"] \\
& & \mathcal{F}\mathrm{Hyp}(\mathcal{C}) \simeq \mathcal{F}\mathrm{Met}(\mathcal{C},\Phi^{[1]})
\end{tikzcd}$$

decalage!

$\rightsquigarrow \mathcal{F}^{hyp}\circ\varrho \longrightarrow 0$ also split. $\square$