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DERIVED ALGEBRAIC COBORDISM

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Introduction

Algebraic cobordism is a generalized cohomology theory for algebraic varieties originally introduced by V. Voevodsky, which is in many ways analogous to complex cobordism in topology. In particular, the special role of complex cobordism in chromatic homotopy theory was a key inspiration for Voevodsky's celebrated proof of the Bloch–Kato conjecture.

In this seminar, we aim to study a new "elementary" construction of algebraic cobordism, due to T. Annala, which uses derived algebraic geometry and is well-behaved over fields of positive characteristic (unlike the previous construction of M. Levine and F. Morel, which strongly relied on resolution of singularities). In addition to general properties such as the bivariant functoriality and the relationship to algebraic K-theory and Chow groups, we will prove the *algebraic Spivak theorem* stating that the derived cobordism groups of a perfect field are generated, up to inverting the characteristic, by cobordism classes of smooth varietes.

Part I: Background in derived algebraic geometry

1. **Derived rings and schemes (26.10).** This talk should introduce the ∞-category of simplicial comutative ring as the nonabelian derived ∞-category of polynomial rings, the ∞-category of derived schemes as topological spaces with sheaves of simplicial commutative rings, and the stable ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on a derived scheme X. There are various references for these definitions, for example [Lur18, Chapter 25], [Kha16, Chapter 0], [AY20, §2]. The following concepts should also be introduced: Noetherian derived schemes, closed immersions and proper morphisms, the classical closed subscheme of a derived scheme, flat morphisms and more generally morphisms locally of Tor-amplitude $\leq n$, and morphisms locally (almost) of finite presentation (e.g., [Lur17], Definitions 7.2.2.10, 7.2.4.21, 7.2.4.26, 7.2.4.30).
2. **Quasi-projective derived schemes (02.11).** Introduce strong sheaves, vector bundles and line bundles on derived schemes [Ann19a, §3]. Define ample line bundle and prove that strong coherent sheaves on quasi-projective derived schemes become globally generated after sufficiently many twists. Prove that in characteristic 0, a derived k-scheme is quasi-projective iff it admits an ample line bundle [Ann19a, §4]. Time permitting, explain the comparison between $K_0(X)$ (defined using perfect complexes) and $K_0^{\mathrm{vect}}(X)$ (defined using vector bundles) [Ann19a, §5], and talk about an example of a non-quasi-projective derived enhancement of a quasi-projective scheme [Ann19c].
3. **The cotangent complex, smoothness, and quasi-smoothness (09.11).** Define derivations of simplicial commutative rings and the (algebraic) cotangent complex $\mathbb{L}_f$ of a morphism of simplicial commutative rings and of derived schemes [Lur18, §25.3], [TV08, §1.2.1]. Define smooth morphisms and quasi-smooth morphisms and explain some of their characterizations, in particular in terms of the cotangent complex [KR19, Propositions 2.3.8, 2.3.14], [TV08, Theorem 2.2.2.6].

4. **Derived blow-ups (16.11).** Construct the blow-up $\mathrm{Bl}_Z(X)$ of a quasi-smooth closed subscheme $Z \subset X$ as the stack of virtual Cartier divisors and show that it is a quasi-smooth derived scheme over X, following Khan and Rydh [KR19]. Introduce also the derived deformation to the normal bundle construction [KR19, §4.1.12]. See also [Ann20, Appendix B].

Part II: The universal property of algebraic cobordism

5. **Bivariant theories (23.11).** Explain the general formalism of bivariant theories and orientations [Ann20, §2.1] and the universal bivariant theory [Ann20, §2.2]. Prove in particular [Ann20, Proposition 2.17], which allows us to construct bivariant theories by imposing relations in homology. Explain the meaning of "confined" and "specialized" for derived schemes [Ann20, Definition 2.22]. Give G_0 and Chow groups as examples of homology theories in this context, and K_0 as an example of a cohomology theory (see also [Ann19b, Definition 3.6]).
6. **The section, formal group law, and strict normal crossings axioms (30.11).** For an oriented bivariant theory on derived schemes, define the first Chern class of a line bundle (more generally, the Euler class of a vector bundle). Then explain the section, fgl and snc axioms and related results [Ann20, §2.3]. Along the way, give a crash course on formal group laws and the Lazard ring.
7. **Precobordism and precobordism with line bundles (07.12).** Define precobordism $\underline{\Omega}^*$ (denoted by $\mathbb{B}^*$ in [AY20]) as the universal naive cobordism theory satisfying the double-point cobordism relation [AY20, Definition 6.1, Proposition 6.3], and cobordism Ω^* as precobordism modulo the snc relation [Ann20, Construction 3.14]. Introduce also the auxiliary variant $\underline{\Omega}^{*,1}$ with line bundles [AY20, Definition 6.6], and prove the structure theorems [AY20, Theorem 6.12, 6.13].
8. **The formal group law of precobordism (14.12)** The main goal of this talk is to construct the formal group law of precobordism, following [AY20, §6.3]. For this one needs the computation $\underline{\Omega}^*(\mathbb{P}^n \times X) \cong \underline{\Omega}^*(X)[t]/(t^{n+1})$, which follows from the weak projective bundle formula [AY20, Theorem 6.22].
9. **The universal property of precobordism and cobordism (21.12).** Prove that $\underline{\Omega}^*$ satisfies the section and formal group law axioms [Ann19b, Theorem 3.4], and that it is universal as a bivariant additive theory with these properties [Ann20, Theorem 3.13] (via [Ann19b, Theorem 3.11]). Explain in particular the proof of [Ann19b, Lemma 3.10], which is a derived version of [LP09, Lemma 5.3], computing the Gysin pushforward of a projectivized line bundle in terms of the formal group law. Deduce that Ω^* is universal with respect to the section, fgl and snc axioms [Ann20, Corollary 3.15].

Part III: Computations of algebraic cobordism

10. **Chern classes and the splitting principle (11.01).** Prove that Euler classes of vector bundles are nilpotent [Ann19b, Lemma 4.2], and explain the construction of Chern classes of vector bundles [Ann19b, §4.1]. Then prove the splitting principle [Ann19b, Theorem 4.11] (via [Ann19b, Theorem 4.8]) and some consequences (the nilpotence of Chern classes and the Whitney sum formula).
11. **The Conner–Floyd and Riemann–Roch theorems (18.01).** State and sketch the proofs of the Conner–Floyd and Riemann–Roch theorems relating $\underline{\Omega}^*$ and K_0 [Ann19b, §5.1,5.2].

12. **The algebraic Spivak theorem (25.01).** Give an overview of this result, which states in particular that for a quasi-projective derived scheme X over a perfect field k of exponential characteristic e, the e-inverted bordism group $\Omega_*(X)[e^{-1}]$ is generated by fundamental classes of smooth k-schemes [Ann21, Theorem 4.12].

References

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