Algebraic Spivak's theorem in positive & mixed characteristic

For us, algebraic Spivak ~ "Algebraic bordism is generated by regular cycles." A <u>regular bordism cycle</u> is one of form $[V \to X]$, where V is regular.

SNC relations

A — regular Noetherian ring, $\dim(A) < \infty$

W — regular, q-proj $/A$

$D = n_1 D_1 + \cdots + n_r D_r$ sncd, i.e.,

for all $I \subset [r]$, $D_I := \bigcap_{i \in I} D_i$ is regular & of expected codim.

Denoting by ι^I the inclusion $D_I \hookrightarrow D$, the SNC relations ask that

$$1_D = \sum_{I \subset [r]} \iota^I_* \left(F_I\left(e(\mathcal{O}(D_1)), \ldots, e(\mathcal{O}(D_r))\right) \bullet 1_{D_I} \right) \in \Omega^A_*(D)$$

where $F_I \in \mathbb{L}[[x_1, \ldots, x_r]]$ depend only on $I \subset [r]$ & $n_1, \ldots, n_r$

Characteristic 0 case (Lowrey – Schürg)

k — field of char 0

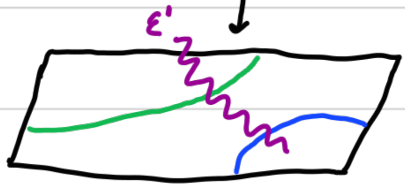
Claim: For X q-projective over k, $\Omega_*^k(X)$ is generated by regular cycles.

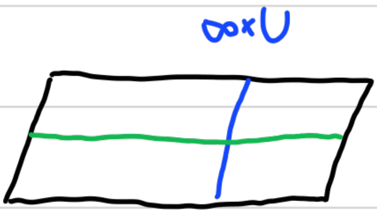
Suffices to show: if Z is q-proj & q-sm, then 1_Z is equiv to an integral comb of regular cycles in $\Omega_*^k(Z)$.

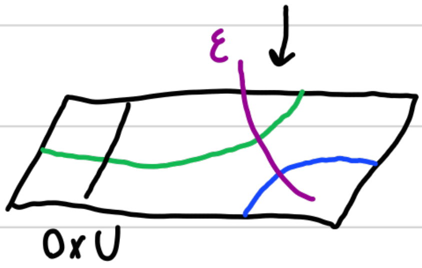
Let $i : Z \hookrightarrow U$ ← sm, q-proj be a derived reg. emb.

$\hat{M}$ is Hironaka res. of sings of M', w/ the inverse image D of $\mathcal{E}'$ sncd & isom / $M' - \mathcal{E}'$.

$\hat{M} :=$ 

$Bl^{cl}_{\infty \times Z_{cl}}(\mathbb{P}^1 \times U) = M' :=$ 

$\mathbb{P}^1 \times U$ 

$\mathbb{P}^1 \times Z \xrightarrow[Z \times \infty]{\text{blow up}} M :=$ 

Considering the Cartesian diagram

$$\begin{tikzcd}
\infty \times Z \arrow[r, hook, "i_\infty"] \arrow[d, hook, "j_\infty"'] & \mathcal{E} = \mathbb{P}_Z(N_{Z/U} \oplus \mathcal{O}) \arrow[l, bend right, "\pi"'] \arrow[d, hook, "\tau_\infty"] \\
\mathbb{P}^1 \times Z \arrow[r, hook, "F"'] & M \\
0 \times Z \arrow[u, hook, "j_0"] \arrow[r, hook, "i"'] & 0 \times U \arrow[u, hook, "\tau_0"']
\end{tikzcd}$$

$$\Rightarrow \quad i^! \circ \tau_0^! = j_0^! \circ F^! = j_\infty^! \circ F^! = i_\infty^! \circ \tau_\infty^!$$

Applying the above formula to $[\check{M} \to M]$, we get

$$\begin{aligned} 1_Z &= i_\infty^!([D \to \mathcal{E}]) \\ &= \pi_*(i_{\infty *}(i_\infty^!([D \to \mathcal{E}]))) \\ &= \pi_*(e(\mathcal{N}_{Z/U}(1)) \bullet [D \to \mathcal{E}]) \end{aligned}$$

$\Rightarrow$ 1_Z is an $\mathbb{Z}$-linear combination of pushforwards of Euler classes on regular schemes.

Bertini + $\Rightarrow$ claim.

<u>Today</u>: generalize the above to positive & mixed char. Hironaka resolution and resolution by alteration. This introduces problems:

i) res by alterations changes the scheme outside the singular locus;

ii) alterations are generically finite $\Rightarrow$ have to invert the res. char. exp. in coeffs.

<u>The main issue we have to deal w/:</u>

$i^!(\tau_0^!([\check{M} \to M]))$ no longer 1_Z. We have to analyze this class in detail.

The setup

A — Henselian & excellent DVR w/ a perfect residue field (or a field)

Z — q-sm & q-proj derived A-scheme

e — the (residual) characteristic exponent

We will show that 1_Z is a $\mathbb{Z}[e^{-1}]$-linear combination of elements of form

$$(\alpha_1 \bullet \cdots \bullet \alpha_s)\,\beta \in \Omega_*^A(Z)[e^{-1}],$$

where $\alpha_i \in \Omega^*(\mathrm{Spec}(A))$ & $\beta \in \Omega_*^A(Z)$ are regular cycles.

Let

$$\begin{tikzcd} Z_{cl} \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{Z}_{cl} \arrow[d, "f"] \\ U \arrow[r, hook] & \mathbb{P}^n_A \end{tikzcd}$$

$$\overline{M}' := Bl^{cl}_{0 \times \overline{Z}_{cl}}(\mathbb{P}^1 \times \mathbb{P}^n_A)$$

$\hat{\overline{M}} \to \overline{M}'$ resolution by alteration (of degree e^r) s.t. the preimages $\overline{W}_0$ & $\overline{W}_\infty$ of $0 \times \mathbb{P}^n_A$ & $\overline{\mathcal{E}}'$, resp., are sncd.

Restrictions $/U$ are denoted $\hat{M}$, W_0, W_∞, $\mathcal{E}'$.

Claim :

$$[\overline{W}_0 \to \mathbb{P}^n_A] = e^r 1_{\mathbb{P}^n_A} + \sum_{i=1}^n \alpha_i \, e(\mathcal{O}(1))^i \cdot 1_{\mathbb{P}^n_A}$$

where α_i are $\mathbb{Z}[e^{-1}]$ - linear combinations of regular cycles.

Pf : Follows from proof of pbf.

We will look into more details if we have the time "$\square$"

Assuming the claim, we can prove Spivak's thm:

i) $i^!([W_0 \to U]) = e^r 1_Z + \sum_{i=1}^n \alpha_i \, e(\mathcal{O}(1)|_Z)^i \cdot 1_Z$

is a $\mathbb{Z}[e^{-1}]$-linear comb of regular cycles.

ii) each $e(\mathcal{O}(1)|_Z)^i \cdot 1_Z$ is equiv to a $\mathbb{Z}[e^{-1}]$-linear combination of $[V \to Z]$, where $\dim(V) < \dim(Z)$

iii) Conclude by induction on Krull dimension. $\square$

Immediate corollaries

Cor: If k a perfect field, X q-proj $/k$,
$\Omega^k_*(X)[e^{-1}]$ is generated, as $\mathbb{Z}[e^{-1}]$-module,
by cycles $[V \to X]$ w/ V is smooth$/k$.

Cor: $\mathbb{L}_*[e^{-1}] \to \Omega^k_*(\mathrm{Spec}(k))[e^{-1}]$ is an isom in degrees ≤ 2.

Levine-Morel's argument: project to a hyp. surf. in $\mathbb{P}^{\leq 3}_k$, use resolution + birational factor.

The extension theorem

Not really an application, but the proof uses similar ideas.

Thm: Let $j: U \hookrightarrow X$ be an open embedding.
Then
$$j^!: \Omega^A_*(X)[e^{-1}] \longrightarrow \Omega^A_*(U)[e^{-1}]$$
is surjective.

If $Z \xhookrightarrow{i} X \xhookleftarrow{j} U$

Want: $\Omega^A_*(Z) \xrightarrow{i_*} \Omega^A_*(X) \xrightarrow{j^!} \Omega^A_*(U) \to 0$ is exact

The extension thm allows us to prove the following version of $\mathbb{A}^1$-invariance.

<u>Cor</u>: Let X be q-proj $/A$ & $p: E \to X$ a vector bundle of rank r. Then

$$p^*: \Omega_*^A(X)[e^{-1}] \to \Omega_{*+r}^A(E)[e^{-1}]$$

is an isomorphism.

$$\mathbb{P}(E) \hookrightarrow \mathbb{P}(E \oplus \mathcal{O}) \hookleftarrow E$$

derived van. loc. of $\mathcal{O}(1)$

$\Rightarrow$ follows from pbf.

Proof of the claim

<u>Lem</u>: Let A be a DVR (or field), X q-proj $/A$. Then for V regular

$$[V \to \mathbb{P}^n \times X] = \sum_{i=0}^{n} e(\mathcal{O}(1))^i \bullet \pi^*(\alpha_i) \in \Omega_*^A(\mathbb{P}^n_* X)$$

where $\alpha_i \in \Omega_*^A(X)$ are integral comb of regular cycles.

Pf: By pbf $\exists! \alpha_i \in \Omega_*^A(X)$ s.t. the equation holds. As

$$\pi_*\left(e(\mathcal{O}(1))^i \bullet [V \to \mathbb{P}^n \times X]\right) = \alpha_{n-i} + [\mathbb{P}_A^1]\alpha_{n-i-1} + \cdots + [\mathbb{P}_A^{n-i}]\alpha_0$$

the claim follows from Bertini+. $\square$

Moreover, if V is flat $/A$, α_i are integral combs of regular cycles flat $/A$ (Bertini+).

Now, if $X = \operatorname{Spec} A$ & $V \to \mathbb{P}_A^n$ is gen. fin. of deg d, then

$$\alpha_0 = \sum_i n_i [\operatorname{Spec}(B_i) \to \operatorname{Spec}(A)] \in \Omega_*^A(\operatorname{Spec}(A))$$

w/ $d = \sum_i n_i \deg(B_i/A)$,

where B_i are reg, flat, integral & finite.

If A is Henselian & has perfect residue field, then B_i/A factors as primitive extensions $\Rightarrow$ $\alpha_0 = d \in \Omega_*^A(\operatorname{Spec}(A))$.

The claim follows from this.