THE COTANGENT COMPLEX, SMOOTHNESS AND QUASI-SMOOTHNESS

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Contents

1. Recollection

Definition 1.1.

- Let Poly denote the category of finitely generated polynomial rings over $\mathbb{Z}$.
- Let PolyMod denote the category of pairs (R, M) where $R \in$ Poly and M is a finitely generated free R-module. A map $(R, M) \to (R', M')$ is a pair (f, φ) where $f\colon R \to R'$ is a map of rings and $\varphi\colon M \to f^*M'$ is a map of R-modules.
- The category of derived rings dRing is the non-abelian derived category of Poly that is dRing $= \mathcal{P}_\Sigma(\text{Poly})$.
- Given a derived ring A, we write dAlg_A for the category of derived A-algebras, i.e., the slice category $\text{dRing}_{A/}$.
- The category dMod^{cn} is the non-abelian derived category of PolyMod. This is the category whose objects are pairs (R, M) where R is a derived ring and M is an R-module.
- Let R be a derived ring. Then the category of connected derived R-modules is defined via the following pullback

$$\begin{array}{ccc} \text{dMod}^{\text{cn}}_R & \longrightarrow & \text{dMod}^{\text{cn}} \\ \downarrow & & \downarrow \\ \{R\} & \longrightarrow & \text{dRing}\,. \end{array}$$

2. Derivations and cotangent complex: affine case

Example 2.1. Let A be a commutative ring and let M be a discrete A-module. Then the direct sum $A \oplus M$ admits the structure of a commutative ring, with multiplication given by

$$(a, m)(a', m') = (aa', am' + a'm).$$

We will refer to $A \oplus M$ as the *trivial square-zero extension of* A by M.

We can extend this construction to the world of derived rings as follows.

Construction 2.2 (Trivial square-zero extension)**.** The construction $(A, M) \mapsto A \oplus M$ determines a functor from PolyMod to the category of commutative rings, which we regard as a full subcategory of dRing. By the theory of non-abelian derived functor, there exists an essentially unique functor $F\colon \text{dMod}^{\text{cn}} \to \text{dRing}$ which commutes with sifted colimits and is given by $(A, M) \mapsto A \oplus M$ on PolyMod. We denote the value of F on a pair (A, M) by $A \oplus M$ and call this the trivial square-zero extension of A by M.

Example 2.3. There is a functor Poly $\to$ PolyMod sending A to $(A, 0)$. Note that the composite

$$\text{Poly} \to \text{PolyMod} \xrightarrow{F} \text{dRing}, \qquad A \mapsto A \oplus 0 = A$$

agrees with the the forgetful functor $U\colon \text{dMod}^{\text{cn}} \to \text{dRing}$ on Poly. By uniqueness of the derived functors we see that F agrees with U on the subcategory of dMod^{cn} spanned by the pairs (A, M) with $M \simeq 0$. Therefore if $M \simeq 0$, then $A \oplus M \simeq A$.

For any A-module M, we have a map $M \to 0$ which induces a map $A \oplus M \to A$. So we can see $A \oplus M$ as an object of $\text{dRing}_{/A}$.

Definition 2.4. Let A be a derived ring and M a connective A-module. We let $\text{Der}(A, M)$ denote the mapping space $\text{Map}_{\text{dRing}_{/A}}(A, A \oplus M)$ and call this the space of derivations of A into M. A derivation is then a morphism $A \to A \oplus M$ together with a commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A \oplus M \\ & {\scriptstyle 1}\searrow \quad \swarrow & \\ & A & \end{array} .$$

The canonical map $0 \to M$ gives a preferred based point for $\text{Der}(A, M)$, the zero derivation.

Example 2.5. Let A be a commutative ring and M an A-module. An element $\phi \in \text{Map}_{\text{Ring}_{/A}}(A, A \oplus M)$ can be thought as a derivation by the following observation. For all $a \in A$, we put $\phi(a) = (a, da) \in A \oplus M$. The condition that ϕ is a ring homomorphism over A forces the relation $d(aa') = a'da + ada'$ and $d1 = 1$.

Lemma 2.6. *Let A be a derived ring. Then there exists a connective A-module L_A and a universal derivation $\eta \in \text{Der}(A, L_A)$ such that evaluation at η gives an equivalence*

$$\text{Map}_{\text{Mod}_A}(L_A, M) \simeq \text{Der}(A, M) = \text{Map}_{\text{dRing}_{/A}}(A, A \oplus M)$$

for any connective A-module M.

Proof. The construction $M \mapsto \text{Der}(A, M)$ determines an accessible functor $\text{Mod}_A^{\text{cn}} \to \mathcal{S}$ which preserves small limits. Therefore it is corepresentable. $\square$

Definition 2.7. The connective A-module L_A is the *cotangent complex* of A.

Example 2.8. Let A be a polynomial ring $\mathbb{Z}[x_s]$ generated by a possibly infinite set of variables $\{x_s\}_{s \in S}$. Let $\Omega_{A/\mathbb{Z}}$ denote the module of Kahler differentials of A: the free A-module generated by the symbols $\{dx_s\}_{s \in S}$. The construction

$$(f \in A) \mapsto (f, \sum \frac{\partial f}{\partial x_s} dx_s)$$

determines a derivation η of A into $\Omega_{A/\mathbb{Z}}$. Moreover, for every connective A-module M, evaluation on η determines a homotopy equivalence

$$\text{Map}_{\text{Mod}_A}(\Omega_{A/\mathbb{Z}}, M) \to \text{Der}(A, M) = \text{Map}_{\text{dRing}_{/A}}(A, A \oplus M)$$

since both sides can be identified with $\prod_{s \in S} \Omega^\infty M$. Therefore η is an universal derivation and $L_A \simeq \Omega_{A/\mathbb{Z}}$; in particular, L_A is a discrete A-module.

There is a functor

$$\text{dRing} \to \text{dMod}^{\text{cn}} \qquad A \mapsto (A, L_A).$$

Definition 2.9. Any map of derived rings $\phi\colon A \to B$ induces a map of B-modules

$$B \otimes_A L_A \to L_B$$

whose cofibre is denoted by $L_{A/B}$ and refer to as the relative algebraic cotangent complex of B over A.

This is another characterization of the relative cotangent complex.

Proposition 2.10. *Let $\phi\colon A \to B$ a morphism of derived rings. For every connective B-module M, we have a canonical equivalence*

$$\text{Map}_{\text{Mod}_B}(L_{B/A}, M) \simeq \text{Map}_{\text{dAlg}_{A/B}}(B, B \oplus M).$$

Proof. There is a fibre sequence

$$\mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, M) \to \mathrm{Map}_{\mathrm{Mod}_B}(L_B, M) \to \mathrm{Map}_{\mathrm{Mod}_B}(B \otimes_A L_A, M).$$

Which we claim we can rewrite as

$$\mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, M) \to \mathrm{Map}_{\mathrm{dRing}_{/B}}(B, B \oplus M) \xrightarrow{\phi^*} \mathrm{Map}_{\mathrm{dRing}_{/B}}(A, B \oplus M)$$

giving the result. The equivalence in the middle term follows by definition of L_B so let us discuss the right most equivalence. We have

$$\mathrm{Map}_{\mathrm{Mod}_B}(B \otimes_A L_A, M) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(L_A, \phi^* M) \simeq \mathrm{Der}(\phi^* M, A) =: \mathrm{Map}_{\mathrm{dRing}_{/A}}(A, A \oplus \phi^* M).$$

Now we note that we have a pullback square of derived rings

$$\begin{array}{ccc} A \oplus \phi^* M & \longrightarrow & B \oplus M \\ \downarrow & & \downarrow \\ A & \xrightarrow{\phi} & B. \end{array}$$

From this description it is clear that

$$\mathrm{Map}_{\mathrm{dRing}_{/A}}(A, A \oplus \phi^* M). \simeq \mathrm{Map}_{\mathrm{dRing}\,/B}(A, B \oplus M).$$

□

Example 2.11. Consider the projection map $A = \mathbb{Z}[x_1, \dots, x_n] \to \mathbb{Z}$ which geometrically corresponds to the inclusion $\{0\} \to \mathbf{A}^n$. Note that $L_{\mathbb{Z}} = 0$ since $\mathbb{Z}$ is initial and L preserves initial objects as it is a left adjoint. Thus $L_{\{0\}/\mathbf{A}^n} = \mathbb{Z} \otimes_A \Omega_{A/\mathbb{Z}}[1] = \mathbb{Z}^n[1]$.

Recall that we have a functor

$$\mathrm{dAlg}_A \to \mathrm{Mod}_A^{\mathrm{cn}} \qquad (\phi\colon A \to B) \mapsto \phi^* B.$$

This has a left adjojnt functor

$$\mathrm{Mod}_A^{\mathrm{cn}} \to \mathrm{dAlg}_A \qquad M \mapsto \mathbb{L}\mathrm{Sym}_A^*(M)$$

the symmetric A-algebra on M.

Example 2.12. Let A be a derived ring and M a connective A-module and put $B = \mathbb{L}\mathrm{Sym}_A^*(M)$ and $\phi\colon A \to B$. We claim that $L_{B/A} \simeq B \otimes_A M$. For any B-module N, we have equivalences

$$\mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, N) \simeq \mathrm{Map}_{\mathrm{dAlg}_{A/B}}(B, B \oplus N) = \mathrm{fib}(\mathrm{Map}_{\mathrm{dAlg}_A}(B, B \oplus N) \to \mathrm{Map}_{\mathrm{dAlg}_A}(B, B))$$

then we use the fact that $\mathbb{L}\mathrm{Sym}_A^*$ is left adjoint so see that

$$\mathrm{Map}_{\mathrm{Mod}_B}(L_{B/A}, N) \simeq \mathrm{fib}(\mathrm{Map}_{\mathrm{Mod}_A}(M, \phi^*(B \oplus N)) \to \mathrm{Map}_{\mathrm{Mod}_A}(M, \phi^* B)) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(M, \phi^* N).$$

Let us list some of the properties that the cotangent complex satisfy.

Proposition 2.13.

(a) *Given a pushout diagram of derived rings*

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \longrightarrow & B \end{array}$$

there is a canonical equivalence $L_{B/A} \simeq B \otimes_{B'} L_{B'/A'}$.

(b) *For every composable pair of morphisms of derived rings* $A \to B \to C$, *there is a canonical cofibre sequence*

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}.$$

(c) *Let* $\varphi\colon A \to B$ *be a morphism of derived rings. Then* φ *is an equivalence iff* $\pi_0(\varphi)$ *is an iso and* $L_{B/A} \simeq 0$.

3. Derivations and cotangent complex: derived schemes

Let $f\colon Y \to X$ be a morphism of derived schemes and let $y\colon \mathrm{Spec}(R) \to Y$ be an R-point. For any $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ we have a commutative square

$$\begin{array}{ccc} Y(R \oplus M) & \xrightarrow{\mathrm{pr}_*} & Y(R) \\ \downarrow & & \downarrow f \\ X(R \oplus M) & \xrightarrow{\mathrm{pr}_*} & X(R) \end{array}$$

where $\mathrm{pr}\colon R \oplus M \to R$ is the canonical projection, which determines a canonical map

$$Y(R \oplus M) \to Y(R) \times_{X(R)} X(R \oplus M).$$

The point $y \in Y(R)$ and the point

$$(d_{triv})_*(f(y))\colon \mathrm{Spec}(R \oplus M) \xrightarrow{d_{triv}} \mathrm{Spec}(R) \xrightarrow{y} Y \xrightarrow{f} X$$

together the canonical isomorphism

$$\mathrm{pr}_*((d_{triv})_*(f(y))) = f(y) \in X(R)$$

defines a point in $Y(R) \times_{X(R)} X(R \oplus M)$. We then define

$$\mathrm{Der}_y(Y/X, M) = \mathrm{fib}(Y(R \oplus M) \to Y(R) \times_{X(R)} X(R \oplus M)).$$

Example 3.1. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ be affine and consider $f\colon \mathrm{Spec}(B) \to \mathrm{Spec}(A)$. The space of derivations at the identity of Y into M is given by the fiber

$$\mathrm{Map}_{\mathrm{dRing}}(B, B \oplus M) \to \mathrm{Map}_{\mathrm{dRing}}(B, B) \times_{\mathrm{Map}_{\mathrm{dRing}}(A,B)} \mathrm{Map}_{\mathrm{dRing}}(A, B \oplus M)$$

which coincides with

$$\mathrm{Map}_{\mathrm{dAlg}_{A/B}}(B, B \oplus M).$$

So $L_{Y/X} = L_{B/A}$ by Proposition 2.10.

Definition 3.2. We say that $L_y \in \mathrm{Mod}_R^{\mathrm{cn}}$ is a cotangent complex of f at y if L_y corepresents the functor $M \mapsto \mathrm{Der}_y(Y/X, M)$ so that there is an equivalence

$$\mathrm{Map}_{\mathrm{Mod}_R}(L_y, M) \simeq \mathrm{Der}_y(Y/X, M).$$

When L_y exists we say that f admits a cotangent complex at y and denote it as $y^*L_{Y/X}$or y^*L_f.

Example 3.3. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ be affine. Then any morphism $f\colon Y \to X$ admits a cotangent complex at any point $y\colon \mathrm{Spec}(R) \to Y$ given by

$$y^*L_{Y/X} = R \otimes_B L_{B/A}$$

Definition 3.4. Let $\mathcal{L}$ be a connective quasi-coherent sheaf on Y. We say that $\mathcal{L}$ is a cotangent complex for $f\colon Y \to X$ if for every points $y \in Y(R)$, the inverse image $y^*\mathcal{L}$ is a cotangent complex of f at y. If $\mathcal{L}$ exists we say that f admits a cotangent complex and write $\mathcal{L}_f$ or $\mathcal{L}_{Y/X}$.

The following properties follow from the definitions.

Proposition 3.5. *Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be morphisms of derived schemes. Suppose that f admits a cotangent complex. Then g admits a cotangent complex if and only if $g \circ f$ does so. In either cases we have a triangle*

$$g^*\mathcal{L}_{Y/X} \to \mathcal{L}_{Z/X} \to \mathcal{L}_{Z/Y}$$

in $\mathrm{Qcoh}(Z)$.

Proposition 3.6. *Let $j\colon U \to X$ be an open immersion of derived schemes. Then j admits a cotangent complex and $\mathcal{L}_{U/X} = 0$.*

Proof. It suffices to show that $u^*\mathcal{L}_{U/X} = 0$ for all points $u \in U(R)$. This amount to the claim that for any $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ the map

$$U(R \oplus M) \to U(R) \times_{X(R)} X(R \oplus M)$$

as contractible fibre at the point $(u, (d_{triv})_*(j(u)))$. Since j is a monomorphism, it is easy to see that this map is also a monomorphism, i.e., it has empty or contractible fibres. Note that the point $(d_{triv})_*(j(u)) \in U(R \oplus M)$ lives in the fibre so we are done. □

Theorem 3.7. *Let $f\colon Y \to X$ be a morphism of derived schemes. Then f admits a cotangent complex $\mathcal{L}_{Y/X} \in \mathrm{Qcoh}(Y)$.*

Proof. If both X and Y admits cotangent complexes over $\mathrm{Spec}(\mathbb{Z})$, denoted $\mathcal{L}_Y$ and $\mathcal{L}_X$ respectively, then we can set

$$\mathcal{L}_{Y/X} = \mathrm{cof}(f^*\mathcal{L}_X \to \mathcal{L}_Y)$$

in view of Proposition 3.5. Therefore we can assume $X = \mathrm{Spec}(\mathbb{Z})$.

Recall that

$$\mathrm{Qcoh}(Y) = \lim_{(U,y)} \mathrm{Qcoh}(U)$$

where the limit is taken over the pairs (U, y) with $U = \mathrm{Spec}(R)$ affine and y an open immersion. Therefore it suffices to construct a compatible system of quasi-coherent sheaves $(y^*\mathcal{L}_Y)_y$ for all such pairs (U, y). According to the exact triangle

$$y^*\mathcal{L}_Y \to \mathcal{L}_U \to \mathcal{L}_{U/Y}$$

and Proposition 3.6, we must have

$$y^*\mathcal{L}_Y \simeq \mathcal{L}_U \tag{1}$$

if $\mathcal{L}_Y$ and $\mathcal{L}_U$ exists. Note that $\mathcal{L}_U$ exists and it coincides with L_R. Given a morphism $j\colon U \to V$ over X we see that $\mathcal{L}_{U/V} \simeq 0$ by the proof of Proposition 3.6 so $j^*L_V \simeq L_U$ showing that this defines an object in the limit and so a quasi-coherent sheaves. □

4. Smoothness and quasi-smoothness

Definition 4.1. Let $\phi\colon A \to B$ be a morphism of derived rings.

- We say that ϕ is *locally of finite presentation* if $\mathrm{Map}_{\mathrm{dAlg}_A}(B, -)$ preserves filtered colimits.
- We say that ϕ is *formally smooth* if the B-module $L_{A/B}$ is projective (=retract of a sum of B's) and if the morphism $B \otimes_A L_A \to L_B$ has a retraction.
- We asy that ϕ is *formally etale* if the morphism $B \otimes_A L_A \to L_B$ is an equivalence, or equivalently $L_{B/A} \simeq 0$.
- We say that ϕ is *smooth* if it is locally of finite presentation and formally smooth.
- We say that ϕ is *etale* if it is locally of finite presentation and formally etale.

Theorem 4.2 (Toen-Vezzosi)**.** *A morphism of derived rings $\phi\colon A \to B$ is etale (resp., smooth) if and only if $\pi_0(B) \otimes_{\pi_0(A)} \pi_*(A) \to \pi_*(B)$ is an isomorphism and moreover the map $\pi_0(A) \to \pi_0(B)$ is etale (resp., smooth).*

Proposition 4.3. *The (formally) smooth and (formally) etale morphisms are stable under composition, pushout and equivalence.*

Proof. This is a consequence of Proposition 2.13. □

Definition 4.4. We say that a morphism of derived schemes $p\colon Y \to X$ is etale (smooth, locally of finite presentation) if there exists Zariski covers $(Y_\alpha \to Y)$ and $(X_\beta \to X)$ such that for each α, there exists β and a morphism of affine derived schemes $Y_\alpha \to X_\beta$ which is etale (smooth, locally of finite presentation) which fit in the following commutative diagram

$$\begin{array}{ccc} Y_\alpha & \longrightarrow & X_\beta \\ \downarrow & & \downarrow \\ X & \longrightarrow & X. \end{array}$$

Proposition 4.5. *Let $p\colon Y \to X$ be a morphism of derived schemes which is locally of finite presentation. Then p is etale (resp., smooth) if and only if the cotangent complex $\mathcal{L}_{Y/X}$ is zero (resp., locally free of finite rank).*

Construction 4.6. Let A be a derived ring and let $f_1, \dots, f_n \in A$ be a sequence of elements. Let $A/\!\!/(f_1, \dots f_n)$ denote the derived ring defined by the pushout

$$\begin{array}{ccc} \mathbb{Z}[x_1, \dots, x_n] & \longrightarrow & \mathbb{Z}[x_1, \dots, x_n]/(x_1, \dots, x_n) \\ \big\downarrow{\scriptstyle x_i \mapsto f_i} & & \big\downarrow \\ A & \longrightarrow & A/\!\!/(f_1, \dots, f_n). \end{array}$$

We have $\pi_0(A/\!\!/(f_1, \dots, f_n)) = \pi_0(A)/(f_1, \dots, f_n)$.

Example 4.7. If A is discrete and the sequence $(f_1, \dots, f_n)$ is regular, then the canonical map $A/\!\!/(f_1, \dots, f_n) \to A/(f_1, \dots, f_n)$ is an isomorphism.

Definition 4.8. Let $i\colon Z \to X$ be a closed immersion of derived schemes (i.e., the underlying morphism of classical scheme is a closed immersion). We say that i is *quasi-smooth* if Zariski locally on X, there exists a morphism $f\colon X \to \mathbf{A}^n$ and a cartensian quare

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \big\downarrow & & \big\downarrow \\ \{0\} & \longrightarrow & \mathbf{A}^n. \end{array}$$

in the category of derived schemes. In other works, i is locally of the form $\mathrm{Spec}(A/\!\!/(f_1, \dots, f_n)) \to \mathrm{Spec}(A)$.

Example 4.9. Let $i\colon Z \to X$ be a closed immersion of classical schemes. Then Zariski locally i is of the form $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ for some ideal I. Then i is regular if and only if I is generated by a regular sequence $f_1, \dots, f_n$. Note that in this case $A/\!\!/(f_1, \dots, d_n) = A/(f_1, \dots, f_n)$. Therefore we deduce that i is regular if and only if it is quasi-smooth.

Proposition 4.10. *Let $i\colon Z \to X$ be a closed immersion of derived schemes. Then i is quasi-smooth if and only if it is locally of finite presentation and the shifted cotangent complex $\mathcal{L}_{Z/X}[1]$ is locally free $\mathcal{O}_Z$-module of finite rank.*

Proof. Suppose that i is quasi-smooth. Since everything is Zariski local and stable under base change, we can assume that i is the inclusion $\{0\} \to \mathbf{A}^n$ in which case $\mathcal{L}_{\{0\}/\mathbf{A}^n}[-1]$ is free of rank n by Example 2.11.

Conversely, suppose that $X = \mathrm{Spec}(A)$ and $Z = \mathrm{Spec}(B)$ are affine, and the shifted cotangent complex $L_{B/A}[-1] \in \mathrm{Mod}_B$ is free of rank n. Let F be the fibre of $\varphi\colon A \to B$ in Mod_A so that there exists a canonical isomorphism of $\pi_0(B)$-module

$$\pi_1(L_{B/A}) = \pi_0(F \otimes_A^L B)$$

induced by Hurewitz map. Choose a basis $df_1, \dots, df_n \in \pi_1(L_{B/A})$ and note that the corresponding elements of $\pi_0(F \otimes_A^L B)$ lift to elements $\tilde{f}_1, \dots, \tilde{f}_n \in \pi_0(F)$ since φ is surjective on π_0. Moreover by Nakayama Lemma we can assume that $\tilde{f}_i$ generates $\pi_0(F)$ as a $\pi_0(A)$-module. Lifting them to points in F, we get points $f_i \in A$ equipped with paths $\varphi(f_i) \simeq 0$ in B. One check that this gives a map $A/\!\!/(f_1, \dots, f_n) \to B$ which by construction is an iso on π_0. To check that this is an isomorphism we can use that the cotangent complex detects isomorphisms. In order words we need to check that the relative cotangent complex vanishes, which follows from the triangle

$$L_{(A/\!\!/(f_i)_i)/A} \otimes_{A/\!\!/(f_i)_i} B \to L_{B/A} \to L_{B/(A/\!\!/(f_i)_i)}.$$

□

Example 4.11. If X and Z are smooth over some base S, then any closed immersion $i\colon Z \to X$ is quasi-smooth. This follows from the exact triangle

$$i^*\mathcal{L}_{X/S} \to \mathcal{L}_{Z/S} \to \mathcal{L}_{Z/X}$$

and Proposition 4.5.

Definition 4.12. A morphism of derived schemes $f\colon Y \to X$ is called *quasi-smooth* if it admits, Zariski locally on Y, a factorization

$$Y \xrightarrow{i} X' \xrightarrow{p} X$$

where i is a quasi-smooth closed immersion and p is smooth.

Recall that a B-module has Tor-amplitude $\leq n$ if for all B-module N, the homotopy groups $\pi_i(M \otimes_B N)$ vanish for $i > n$.

Proposition 4.13. *Let $f\colon Y \to X$ a morphism of derived schemes. Then f is quasi-smooth if and only if f is locally of finite presentation and the cotangent complex $\mathcal{L}_{Y/X}$ is of Tor-amplitude ≤ 1.*

Proof. Lurie tells us that the question is local on Y. If f admits a factorization as above, then f is locally of finite presentation. The exact triangle

$$i^*\mathcal{L}_{X'/X} \to \mathcal{L}_{Y/X} \to \mathcal{L}_{Y/X'}$$

shows that $\mathcal{L}_{Y/X}$ is also of Tor-amplitude ≤ 1.

For the converse direction, let $\varphi\colon A \to B$ be a morphism of derived ring which is locally of finite presentation and $L_{B/A}$ has Tor-amplitude ≤ 1. Then $\pi_0(A) \to \pi_0(B)$ is of finite presentation so we can find a homomorphism $A' = A[x_1, \dots, x_n] \to B$ that extends φ and is surjective on π_0. The homomorphism $A \to A'$ is smooth so it will suffices to show that $L_{B/A'}[-1]$ is locally free of finite rank. Or equivalently, since $\pi_0(L_{B/A'}) = \Omega^1_{\pi_0(B)/\pi_0(A')} = 0$ that $L_{B/A'}$ is of Tor-amplitude ≤ 1. This follows from the triangle

$$L_{A'/A} \otimes_{A'} B \to L_{B/A} \to L_{B/A'}.$$

$\square$