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Derived blow-ups

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Novemeber 16, 2021

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Overview



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Recollection on quasi-smoothness

Fix a morphism $f : Z \to X$ of derived schemes.

• If f is a closed immersion, then it is quasi-smooth if, Zariski-locally on X, it is of the form

$$\operatorname{Spec}(A//(f_1,\ldots,f_n)) \to \operatorname{Spec}(A)$$

for certain points $f_1, \ldots, f_n \in A$. That is, Zariski-locally on X we have Cartesian diagrams in dSch



• In general, *f* is quasi-smooth if it admits, Zariski-locally on *Z*, a factorization

$$Z \xrightarrow{i} X' \xrightarrow{p} X$$

where *i* is a quasi-smooth closed immersion, and *p* is smooth.

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Recollection on cotangent bundle

• The cotangent complex of *f* is the connective, quasi-coherent \mathcal{O}_Z -module with the universal property that

 $\operatorname{QCoh}(\mathcal{T})(g^*\mathcal{L}_{Z/X},\mathcal{M})\simeq\operatorname{Sch}_{\mathcal{T}/Y}(\operatorname{Spec}(\mathcal{O}_{\mathcal{T}}\oplus\mathcal{M}),Z)$

for any $g: T \to Z$ and any $\mathcal{M} \in \operatorname{QCoh}(T)^{\operatorname{cn}}$.

- Define the normal sheaf as $\mathcal{N}_{Z/X} := \mathcal{L}_{Z/X}[-1].$
- If f is a closed immersion, then it is quasi-smooth if and only if it is locally of finite presentation and $\mathcal{N}_{Z/X}$ is locally free of finite rank. Then define the virtual codimension of Z in X as the rank of $\mathcal{N}_{Z/X}$.

Example

The virtual codimension of $\text{Spec}(A//(f_1, \ldots, f_n)) \rightarrow \text{Spec } A$ is n. If A is discrete and f_1, \ldots, f_n is a regular sequence, then $\mathcal{N}_{Z/X} \simeq I/I^2$, where $I = (f_1, \ldots, f_n)$.

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Virtual Cartier divisors

Throughout, let $i : Z \rightarrow X$ be a quasi-smooth closed immersion of derived schemes.

Definition

- A virtual Cartier divisor on a derived scheme S is a quasi-smooth closed immersion $D \rightarrow S$ of virtual codimension 1.
- For $S \to X$, a virtual Cartier divisor on S over (X, Z) is a commutative diagram

$$D \longrightarrow S$$

$$\downarrow^{g} \qquad \downarrow$$

$$Z \xrightarrow{i} X$$

in $\operatorname{dSch}\nolimits$, such that

- D
 ightarrow S is a virtual Cartier divisor
- The underlying square of classical schemes is Cartesian
- The induced map $g^*\mathcal{N}_{Z/X} o \mathcal{N}_{D/S}$ is surjective on π_0 .

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Alternative characterization

Remark

For $S \to X$, put $S_Z := S \times_X Z$. Then a virtual Cartier divisor over (X, Z) on S is equivalently

- A derived schemed $h: D \rightarrow S_Z$,
- ullet such that D
 ightarrow S is a virtual Cartier divisor, and
- the map $(D)_{
 m cl}
 ightarrow (S_Z)_{
 m cl}$ is an isomorphism, and
- the induced map $h^* \mathcal{N}_{S_Z/S} \to \mathcal{N}_{D/S}$ is surjective on π_0 .

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The space of virtual Cartier divisors

Definition

For a derived scheme S over X, we write $BI_Z(X)(S)$ for the space of virtual Cartier divisors on S lying over (X, Z). Since virtual Cartier divisors over (X, Z) are stable under pullbacks, this gives a functor

$$\mathsf{Bl}_Z X : \mathrm{dSch}_{/X}^{\mathrm{op}} \to \mathbb{S}$$

This is a stack for the étale topology, called the blow-up.

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Existence

Theorem

The blow-up functor $BI_Z X$ is represented by a derived scheme over X.

Two main ingredients:

Lemma

The blow-up functor commutes with derived base-change: for any $X' \to X$ we have $(Bl_Z X) \times_X X' \simeq Bl_{Z \times_X X'} X'$.

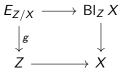
Lemma

The functor $BI_{\{0\}} \mathbb{A}^n$ is representable.

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The universal virtual Cartier divisor

The identity $BI_Z X \rightarrow BI_Z X$ induces a virtual Cartier divisor over (X, Z)



The scheme $E_{Z/X}$ is the exceptional divisor.

Remark

- Recall $\operatorname{Map}_{Z}(E_{Z/X}, \mathbb{P}(\mathcal{N}_{Z/X}))$ is the space of pairs $(g^*\mathcal{N}_{Z/X} \twoheadrightarrow \mathcal{L}, \mathcal{L} \in \operatorname{Pic}(E_{Z/X})).$
- $\mathcal{N}_{E_{Z/X}/\operatorname{Bl}_{Z/X}}$ is a line bundle on $E_{Z/X}$.
- The induced map $E_{Z/X} \to \mathbb{P}(\mathcal{N}_{Z/X})$ is an equivalence.

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Main properties

Definition

A morphism f of derived schemes is proper if f_{cl} is proper.

Theorem

- The structure morphism $\pi : BI_Z X \to X$ is proper and quasi-smooth.
- π induces an equivalence $\operatorname{Bl}_Z X \setminus E_{Z/X} \to X \setminus Z$.
- If X is quasi-projective over a Noetherian ring, then π is projective.
- If X, Z are both classical, then Bl_Z X is the classical blow-up of X in Z.

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Strict transforms

Let $Z \to X \xrightarrow{f} Y$ be quasi-smooth closed immersions in dSch.

Proposition

- $E_{Z/X} \rightarrow Bl_Z X$ induces a virtual Cartier divisor over (Y, Z).
- The induced morphism Bl_Z X → Bl_Z Y is a quasi-smooth closed immersion, called the strict transform of X → Y.

Proof of first point.

It suffices to show that a virtual Cartier divisor $D \rightarrow S$ over $Z \rightarrow X$ induces one on $Z \rightarrow Y$. Write $g : D \rightarrow Z$ for the structure map.

- Since $Z_{cl} = (Z \times_Y X)_{cl}$, it holds $(D)_{cl} \cong (Z \times_X S)_{cl} \cong (Z \times_Y S)_{cl}$.
- In the fiber sequence $g^*f^*\mathcal{N}_{X/Y} \to g^*\mathcal{N}_{Z/Y} \to g^*\mathcal{N}_{Z/X}$, the last map is surjective on π_0 . Hence so is

$$g^* \mathcal{N}_{Z/Y} o g^* \mathcal{N}_{Z/X} o \mathcal{N}_{D/S}$$

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Strict transforms

Corollary

We have a Cartesian diagram in dSch

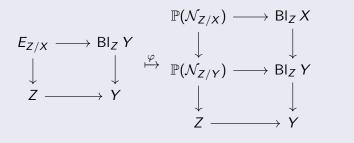
where the top arrow is induced by the surjection $\mathcal{N}_{Z/Y} \rightarrow \mathcal{N}_{Z/X}$.

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Proof.					

The strict transform $t : Bl_Z X \rightarrow Bl_Z Y$ induced a map of spaces

 $\varphi: \operatorname{Bl}_Z(Y)(\operatorname{Bl}_Z Y) \to \operatorname{Bl}_Z Y(\operatorname{Bl}_Z X)$

which sends the identity to the virtual Cartier divisor $E_{Z/X} \rightarrow BI_Z X$ over $Z \rightarrow Y$.On the level of virtual Cartier divisors, this is



now the claim follows by definition of $BI_Z(Y)(-)$.

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Blow-ups in multiple centres

Proposition

Let $Z_1 \to X, Z_2 \to X$ be quasi-smooth closed immersions in dSch. Write $W = Z_1 \times_X Z_2$.

- The exceptional divisor of Bl_W X is P(N_{Z1/X|W} ⊕ N_{Z2/X|W}). It meets the strict transform Bl_W Z_i → Bl_W X in P(N_{W/Zi}).
- The strict transforms Bl_W Z₁ → Bl_W X and Bl_W Z₂ → Bl_W X do not meet inside Bl_W X.

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Weil restrictions

Now let $i : Z \to X$ be any closed immersion of derived schemes.

Definition

Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an affine morphism of stacks. Then $f^* : \operatorname{St}_{/\mathfrak{Y}} \to \operatorname{St}_{/\mathfrak{X}}$ has a right adjoint. This is called the Weil restriction, written Res_f .

Consider the map $\zeta : B\mathbb{G}_{m,X} \to [\mathbb{A}^1_X/\mathbb{G}_{m,X}]$ induced by the zero section.

Definition

Define the $\mathbb{G}_{m,X}$ -stack $D_{Z/X}$ via the pullback

$$\begin{array}{ccc} D_{Z/X} & \longrightarrow \operatorname{Res}_{\zeta}[Z/\mathbb{G}_{m,X}] \\ & \downarrow & & \downarrow \\ \mathbb{A}^1_X & \longrightarrow [\mathbb{A}^1_X/\mathbb{G}_{m,X}] \end{array}$$

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Derived blow-ups via Weil restrictions

Theorem

The stack $D_{Z/X}$ is representable by a derived scheme over \mathbb{A}^1_X . In fact, the structure map $D_{Z/X} \to \mathbb{A}^1_X$ is affine.

Definition

The associated \mathbb{Z} -graded $\mathcal{O}_X[t^{-1}]$ -algebra is the derived extended Rees algebra of $Z \to X$, written $\mathcal{R}_{Z/X}^{\text{ext}}$.

Definition

Now define the derived blow-up as $BI_Z X := \operatorname{Proj} \mathcal{R}_{Z/X}$.

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Main properties

Theorem

- If Z → X is quasi-smooth, then this construction of Bl_Z X coincides with the previous one.
- Bl_Z X is stable under pullback.
- We have a strict transform $BI_Z X \to BI_Z Y$ for any closed immersion $X \to Y$.
- For A → B a surjection, the extended Rees algebra satisfies the universal property that

$$\mathcal{A} \mathrm{lg}_{\mathcal{A}[t^{-1}]}^{\mathbb{Z}}(\mathcal{R}^{\mathsf{ext}}_{B/A}, Q) \simeq \mathcal{A} \mathrm{lg}_{\mathcal{A}}(B, (Q/(t^{-1}))_0)$$

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Main properties

Theorem

• The universal property induces an adjunction

$$R_{(-)/A}^{\operatorname{ext}}$$
: $\operatorname{Sur}_A \rightleftharpoons \operatorname{Rees}_A$: G_A

where $G_A(Q) := (Q/(t^{-1}))_0$. The left adjoint $R_{(-)/A}^{\text{ext}}$ is fully faithful.

• For $Z \to X$ classical, we recover the classical extended Rees algebra as $(\pi_0 \mathcal{R}_{Z/X}^{\text{ext}})^{\text{reg}}$.

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Thank you!

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