Derived blow-ups

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Overview

Recollection on quasi-smoothness

Fix a morphism $f : Z \to X$ of derived schemes.

- If f is a closed immersion, then it is quasi-smooth if, Zariski-locally on X, it is of the form

$$\mathrm{Spec}(A//(f_1, \ldots, f_n)) \to \mathrm{Spec}(A)$$

for certain points $f_1, \ldots, f_n \in A$. That is, Zariski-locally on X we have Cartesian diagrams in $\mathrm{d}\mathcal{S}\mathrm{ch}$

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^n \end{array}$$

- In general, f is quasi-smooth if it admits, Zariski-locally on Z, a factorization

$$Z \xrightarrow{i} X' \xrightarrow{p} X$$

where i is a quasi-smooth closed immersion, and p is smooth.

Recollection on cotangent bundle

- The cotangent complex of f is the connective, quasi-coherent $\mathcal{O}_Z$-module with the universal property that

$$\mathrm{QCoh}(T)(g^*\mathcal{L}_{Z/X}, \mathcal{M}) \simeq \mathrm{Sch}_{T/Y}(\mathrm{Spec}(\mathcal{O}_T \oplus \mathcal{M}), Z)$$

  for any $g : T \to Z$ and any $\mathcal{M} \in \mathrm{QCoh}(T)^{\mathrm{cn}}$.
- Define the normal sheaf as $\mathcal{N}_{Z/X} := \mathcal{L}_{Z/X}[-1]$.
- If f is a closed immersion, then it is quasi-smooth if and only if it is locally of finite presentation and $\mathcal{N}_{Z/X}$ is locally free of finite rank. Then define the virtual codimension of Z in X as the rank of $\mathcal{N}_{Z/X}$.

Example

The virtual codimension of $\mathrm{Spec}(A//(f_1, \ldots, f_n)) \to \mathrm{Spec}\, A$ is n. If A is discrete and $f_1, \ldots, f_n$ is a regular sequence, then $\mathcal{N}_{Z/X} \simeq I/I^2$, where $I = (f_1, \ldots, f_n)$.

Virtual Cartier divisors

Throughout, let $i : Z \to X$ be a quasi-smooth closed immersion of derived schemes.

Definition

- A virtual Cartier divisor on a derived scheme S is a quasi-smooth closed immersion $D \to S$ of virtual codimension 1.
- For $S \to X$, a virtual Cartier divisor on S over (X, Z) is a commutative diagram

$$\begin{array}{ccc} D & \longrightarrow & S \\ \big\downarrow{\scriptstyle g} & & \big\downarrow \\ Z & \xrightarrow{\ i\ } & X \end{array}$$

  in $\mathrm{d}\mathcal{S}\mathrm{ch}$, such that
  - $D \to S$ is a virtual Cartier divisor
  - The underlying square of classical schemes is Cartesian
  - The induced map $g^*\mathcal{N}_{Z/X} \to \mathcal{N}_{D/S}$ is surjective on π_0.

Alternative characterization

Remark

For $S \to X$, put $S_Z := S \times_X Z$. Then a virtual Cartier divisor over (X, Z) on S is equivalently

- A derived schemed $h : D \to S_Z$,
- such that $D \to S$ is a virtual Cartier divisor, and
- the map $(D)_{\mathrm{cl}} \to (S_Z)_{\mathrm{cl}}$ is an isomorphism, and
- the induced map $h^* \mathcal{N}_{S_Z/S} \to \mathcal{N}_{D/S}$ is surjective on π_0.

The space of virtual Cartier divisors

Definition

For a derived scheme S over X, we write $\mathrm{Bl}_Z(X)(S)$ for the space of virtual Cartier divisors on S lying over (X, Z). Since virtual Cartier divisors over (X, Z) are stable under pullbacks, this gives a functor

$$\mathrm{Bl}_Z X : \mathrm{d}\mathcal{S}\mathrm{ch}_{/X}^{\mathrm{op}} \to \mathcal{S}$$

This is a stack for the étale topology, called the blow-up.

Existence

Theorem

The blow-up functor $\mathrm{Bl}_Z X$ *is represented by a derived scheme over* X.

Two main ingredients:

Lemma

The blow-up functor commutes with derived base-change: for any $X' \to X$ *we have* $(\mathrm{Bl}_Z X) \times_X X' \simeq \mathrm{Bl}_{Z \times_X X'} X'$.

Lemma

The functor $\mathrm{Bl}_{\{0\}} \mathbb{A}^n$ *is representable.*

The universal virtual Cartier divisor

The identity $\mathrm{Bl}_Z X \to \mathrm{Bl}_Z X$ induces a virtual Cartier divisor over (X, Z)

$$\begin{array}{ccc} E_{Z/X} & \longrightarrow & \mathrm{Bl}_Z X \\ \big\downarrow{\scriptstyle g} & & \big\downarrow \\ Z & \longrightarrow & X \end{array}$$

The scheme $E_{Z/X}$ is the exceptional divisor.

Remark

- Recall $\mathrm{Map}_Z(E_{Z/X}, \mathbb{P}(\mathcal{N}_{Z/X}))$ is the space of pairs $(g^*\mathcal{N}_{Z/X} \twoheadrightarrow \mathcal{L}, \mathcal{L} \in \mathcal{P}\mathrm{ic}(E_{Z/X}))$.
- $\mathcal{N}_{E_{Z/X}/\mathrm{Bl}_{Z/X}}$ is a line bundle on $E_{Z/X}$.
- The induced map $E_{Z/X} \to \mathbb{P}(\mathcal{N}_{Z/X})$ is an equivalence.

Main properties

Definition

A morphism f of derived schemes is proper if f_{cl} is proper.

Theorem

- *The structure morphism* $\pi : \mathrm{Bl}_Z X \to X$ *is proper and quasi-smooth.*
- π *induces an equivalence* $\mathrm{Bl}_Z X \setminus E_{Z/X} \to X \setminus Z$*.*
- *If* X *is quasi-projective over a Noetherian ring, then* π *is projective.*
- *If* X, Z *are both classical, then* $\mathrm{Bl}_Z X$ *is the classical blow-up of* X *in* Z*.*

Strict transforms

Let $Z \to X \xrightarrow{f} Y$ be quasi-smooth closed immersions in $\mathrm{d}\mathcal{S}\mathrm{ch}$.

Proposition

- *$E_{Z/X} \to \mathrm{Bl}_Z X$ induces a virtual Cartier divisor over (Y, Z).*
- *The induced morphism $\mathrm{Bl}_Z X \to \mathrm{Bl}_Z Y$ is a quasi-smooth closed immersion, called the strict transform of $X \to Y$.*

Proof of first point.

It suffices to show that a virtual Cartier divisor $D \to S$ over $Z \to X$ induces one on $Z \to Y$. Write $g : D \to Z$ for the structure map.

- Since $Z_{\mathrm{cl}} = (Z \times_Y X)_{\mathrm{cl}}$, it holds $(D)_{\mathrm{cl}} \cong (Z \times_X S)_{\mathrm{cl}} \cong (Z \times_Y S)_{\mathrm{cl}}$.
- In the fiber sequence $g^* f^* \mathcal{N}_{X/Y} \to g^* \mathcal{N}_{Z/Y} \to g^* \mathcal{N}_{Z/X}$, the last map is surjective on π_0. Hence so is

$$g^* \mathcal{N}_{Z/Y} \to g^* \mathcal{N}_{Z/X} \to \mathcal{N}_{D/S}$$

Strict transforms

Corollary

We have a Cartesian diagram in $\mathrm{d}\mathcal{S}\mathrm{ch}$

$$\begin{array}{ccc} \mathbb{P}(\mathcal{N}_{Z/X}) & \longrightarrow & \mathbb{P}(\mathcal{N}_{Z/Y}) \\ \downarrow & & \downarrow \\ \mathrm{Bl}_Z X & \longrightarrow & \mathrm{Bl}_Z Y \end{array}$$

where the top arrow is induced by the surjection $\mathcal{N}_{Z/Y} \to \mathcal{N}_{Z/X}$.

Proof.

The strict transform $t : \mathrm{Bl}_Z X \to \mathrm{Bl}_Z Y$ induced a map of spaces

$$\varphi : \mathrm{Bl}_Z(Y)(\mathrm{Bl}_Z Y) \to \mathrm{Bl}_Z Y(\mathrm{Bl}_Z X)$$

which sends the identity to the virtual Cartier divisor $E_{Z/X} \to \mathrm{Bl}_Z X$ over $Z \to Y$.On the level of virtual Cartier divisors, this is

$$\begin{array}{ccc} E_{Z/X} & \longrightarrow & \mathrm{Bl}_Z Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array} \quad \overset{\varphi}{\mapsto} \quad \begin{array}{ccc} \mathbb{P}(\mathcal{N}_{Z/X}) & \longrightarrow & \mathrm{Bl}_Z X \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{N}_{Z/Y}) & \longrightarrow & \mathrm{Bl}_Z Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Y \end{array}$$

now the claim follows by definition of $\mathrm{Bl}_Z(Y)(-)$. □

Blow-ups in multiple centres

Proposition

Let $Z_1 \to X, Z_2 \to X$ *be quasi-smooth closed immersions in* $\mathrm{d}\mathcal{S}\mathrm{ch}$*. Write* $W = Z_1 \times_X Z_2$*.*

- *The exceptional divisor of* $\mathrm{Bl}_W X$ *is* $\mathbb{P}(\mathcal{N}_{Z_1/X|W} \oplus \mathcal{N}_{Z_2/X|W})$*. It meets the strict transform* $\mathrm{Bl}_W Z_i \to \mathrm{Bl}_W X$ *in* $\mathbb{P}(\mathcal{N}_{W/Z_i})$*.*
- *The strict transforms* $\mathrm{Bl}_W Z_1 \to \mathrm{Bl}_W X$ *and* $\mathrm{Bl}_W Z_2 \to \mathrm{Bl}_W X$ *do not meet inside* $\mathrm{Bl}_W X$*.*

Weil restrictions

Now let $i : Z \to X$ be any closed immersion of derived schemes.

Definition

Let $f : \mathcal{X} \to \mathcal{Y}$ be an affine morphism of stacks. Then $f^* : \mathcal{S}\mathrm{t}_{/\mathcal{Y}} \to \mathcal{S}\mathrm{t}_{/\mathcal{X}}$ has a right adjoint. This is called the Weil restriction, written Res_f.

Consider the map $\zeta : B\mathbb{G}_{m,X} \to [\mathbb{A}^1_X/\mathbb{G}_{m,X}]$ induced by the zero section.

Definition

Define the $\mathbb{G}_{m,X}$-stack $D_{Z/X}$ via the pullback

$$\begin{array}{ccc} D_{Z/X} & \longrightarrow & \mathrm{Res}_\zeta[Z/\mathbb{G}_{m,X}] \\ \downarrow & & \downarrow \\ \mathbb{A}^1_X & \longrightarrow & [\mathbb{A}^1_X/\mathbb{G}_{m,X}] \end{array}$$

Derived blow-ups via Weil restrictions

Theorem

The stack $D_{Z/X}$ is representable by a derived scheme over $\mathbb{A}^1_X$. In fact, the structure map $D_{Z/X} \to \mathbb{A}^1_X$ is affine.

Definition

The associated $\mathbb{Z}$-graded $\mathcal{O}_X[t^{-1}]$-algebra is the derived extended Rees algebra of $Z \to X$, written $\mathcal{R}^{\text{ext}}_{Z/X}$.

Definition

Now define the derived blow-up as $\text{Bl}_Z X := \text{Proj}\, \mathcal{R}_{Z/X}$.

Main properties

Theorem

- *If $Z \to X$ is quasi-smooth, then this construction of* $\mathrm{Bl}_Z X$ *coincides with the previous one.*
- $\mathrm{Bl}_Z X$ *is stable under pullback.*
- *We have a strict transform* $\mathrm{Bl}_Z X \to \mathrm{Bl}_Z Y$ *for any closed immersion* $X \to Y$.
- *For $A \to B$ a surjection, the extended Rees algebra satisfies the universal property that*

$$\mathcal{A}\mathrm{lg}^{\mathbb{Z}}_{A[t^{-1}]}(R^{\mathrm{ext}}_{B/A}, Q) \simeq \mathcal{A}\mathrm{lg}_A(B, (Q/(t^{-1}))_0)$$

Main properties

Theorem

- *The universal property induces an adjunction*

$$R^{\text{ext}}_{(-)/A} : \mathcal{S}\mathrm{ur}_A \rightleftarrows \mathcal{R}\mathrm{ees}_A : G_A$$

where $G_A(Q) := (Q/(t^{-1}))_0$*. The left adjoint* $R^{\text{ext}}_{(-)/A}$ *is fully faithful.*

- *For* $Z \to X$ *classical, we recover the classical extended Rees algebra as* $(\pi_0 \mathcal{R}^{\text{ext}}_{Z/X})^{\text{reg}}$*.*

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Thank you!