WISE 2021/22 SEMINAR: INTRODUCTION TO STABLE HOMOTOPY THEORY

Originally a subfield of algebraic topology that emerged in the second half of the 20th century, stable homotopy theory nowadays plays a much larger role in mathematics and has applications to various fields such as geometric topology, algebraic geometry, and even number theory.

The goal of this seminar is to introduce the central notion of spectrum and study its basic properties. Among other things, we will discuss the equivalence between spectra and generalised cohomology theories, the smash product of spectra, Spanier–Whitehead duality, Atiyah duality, the Steenrod algebra, the Atiyah–Hirzebruch and Adams spectral sequences, and the relationship between stable homotopy and bordism of smooth manifolds.

Recommended prerequisites: Algebraic Topology I and II (in particular: homology groups, homotopy groups, CW complexes, Eilenberg–Mac Lane spaces)

List of talks

(1) **Introduction**

Part 1. Generalities on spectra

(2) **Spectra: definitions and examples**
Define spectra and Ω-spectra following [Ada74, §2]. Explain how generalised cohomology theories give rise to Ω-spectra via Brown representability. Discuss the following examples of spectra: the Eilenberg–Mac Lane spectrum HA of an abelian group A; the 2-periodic complex K-theory spectrum KU; the suspension spectrum $\Sigma^\infty X$ of a pointed space X, in particular the sphere spectrum $\mathbb{S} = \Sigma^\infty S^0$; the Thom spectrum MO. Define the homotopy groups $\pi_*(E)$ of a spectrum E.

(3) **The homotopy category of spectra I: Construction**
Define CW spectra, morphisms between (CW) spectra and homotopies between them [Ada74, pp. 139–144]. The *homotopy category of spectra* has objects the CW spectra and morphisms the homotopy classes of morphisms (of degree 0). Show that $[\Sigma^\infty X, E] \cong \operatorname{colim}_{n\to\infty}[\Sigma^n X, E_n]$ [Ada74, Proposition 2.8]. Define the "stable cells" of a CW spectrum [Ada74, §3]. Show that homotopy groups detect isomorphisms in the homotopy category of spectra [Ada74, Corollary 3.5], and that every CW spectrum is homotopy equivalent to an Ω-spectrum.

(4) **The homotopy category of spectra II: Properties**
Show that suspension induces a self-equivalence of the stable homotopy category [Ada74, Theorem 3.7] Define cofibre sequences of spectra and construct the associated long exact sequences [Ada74, Propositions 3.9 and 3.10]. State the variant of Brown's representability theorem for CW spectra [Ada74, Theorem 3.12], and deduce that every spectrum is weakly equivalent to a CW spectrum. Show that the homotopy category of spectra admits arbitrary sums and products, and is additive (i.e., it has a zero object 0 and finite sums and products coincide) [Ada74, Proposition 3.14].

(5) **The smash product**
The homotopy category of spectra has a symmetric monoidal structure given by the "smash product" $\wedge$. Its construction is notoriously technical, but having a symmetric monoidal structure on the homotopy category of spectra is absolutely essential for most applications. State [Ada74, Theorem 4.1] summarizing the existence and properties of the smash product (familiarity with the notion of symmetric monoidal category may be assumed). Define the "naive smash product" $X \wedge_{BC} Y$ and explain its eventual relation to $X \wedge Y$ [Ada74, Theorem 4.2]. Finally, sketch the actual construction of $X \wedge Y$ using a "double telescope", following [Ada74, §4]. This construction is also sketched in [Swi75, Chapter 13].

(6) **Homology, cohomology, and products**
For a spectrum E, define the E-homology and E-cohomology groups of a spectrum and of a space [Ada74, §6], as well as the reduced/relative versions. Explain the long exact sequence of a triple and the Mayer–Vietoris sequence. Define the general external products and slant products in homology and cohomology [Ada74, beginning of §9]. Deduce that $E^*(X)$ is a graded

(commutative) ring when E is a (commutative) ring spectrum and X is a space, and that $E_*(X)$ is an $E^*(X)$-modules, cf. [Ada74, §9, "Internal Products"].

(7) **Derived inverse limits and the Milnor exact sequence**
For a spectrum E and a CW complex X union of a sequence of subcomplexes X_n, the homology groups $E_*(X)$ are the colimit of the homology groups $E_*(X_n)$. For cohomology, which is a contravariant functor, the situation is more complicated. In general, there is the *Milnor exact sequence*

$$0 \to \lim^1_n E^{*-1}(X_n) \to E^*(X) \to \lim_n E^*(X_n) \to 0.$$

Review the construction of $\lim^1$ of a sequence of abelian groups, and derive the Milnor exact sequence [Ada74, Proposition 8.1]. Illustrate with some examples (e.g., the ordinary cohomology and K-theory of $\mathbb{C}P^\infty$).

(8) **Spanier–Whitehead duality**
Spanier–Whitehead duality refers to the fact that the homotopy category of finite CW spectra is self-dual, i.e., equivalent to its opposite. Review the notion of dualisable object in a (symmetric) monoidal category (see e.g. [DP84, §1]). Define the dual X^* of a CW spectrum X and the evaluation map $e\colon X^* \wedge X \to \mathbb{S}$ using Brown representability [Ada74, §5]. When X is finite, show that X^* is finite [Ada74, Lemma 5.5, p. 199] and that e exhibits X as a dualisable object with dual X^* [Ada74, Proposition 5.4]. Discuss the connection with Alexander duality for compact (locally contractible) subspaces of S^n.

(9) **Atiyah duality**
Atiyah duality is a generalisation of Poincaré duality that applies to arbitrary generalised cohomology theories. It is the statement that the suspension spectrum $\Sigma^\infty M_+$ of a closed smooth manifold M is dualisable in the stable homotopy category, with dual the Thom spectrum of the stable normal bundle of M. Give the proof of this result, following Dold and Puppe [DP84, Theorem 3.1]. Explain also how to obtain the classical Poincaré duality as a special case.

Part 2. Computations

(10) **Spectral sequences**
Introduce exact couples and spectral sequences, discussing in detail the primary example of the spectral sequence of a filtered chain complex, and in which sense this spectral sequence converges for a finite filtration. There are many references for these topics, for instance [HS97, VIII, §1–3] and [Boa99].

(11) **The Atiyah–Hirzebruch spectral sequence**
The construction of the exact couple of a filtered chain complex in the previous talk yields in exactly the same way an exact couple for the E-homology or E-cohomology of a filtered *spectrum*. Construct the Atiyah–Hirzebruch spectral sequence

$$E^2_{p,q} = H_p(X, \pi_q E) \Rightarrow E_{p+q}(X)$$

and its cohomological version as a special case of this observation, where X is a finite-dimensional CW complex [Ada74, §7]. Work through the example of the complex K-theory of $\mathbb{C}P^n$. Discuss the question of convergence when X is infinite-dimensional [Ada74, Theorem 8.2].

(12) **The Steenrod algebra and its dual**
The goal of this talk is to present the structure of the Steenrod algebra $\mathcal{A}^* = (H\mathbb{Z}/p)^*(H\mathbb{Z}/p)$ and its dual $\mathcal{A}_* = (H\mathbb{Z}/p)_*(H\mathbb{Z}/p)$, for simplicity at the prime $p = 2$. These are Hopf algebras over $\mathbb{Z}/p$ which play a crucial role in homotopy theory. For every space X, $H^*(X, \mathbb{Z}/p)$ is an $\mathcal{A}^*$-module and $H_*(X, \mathbb{Z}/p)$ is an $\mathcal{A}_*$-comodule. First, sketch the construction of the Steenrod squares $\mathrm{Sq}^i\colon H^*(X, \mathbb{Z}/2) \to H^{*+i}(X, \mathbb{Z}/2)$, following [Swi75, Chapter 18], and state the structure of $\mathcal{A}^*$ as a $\mathbb{Z}/p$-algebra generated by the Steenrod squares modulo the Adem relations. Then define the elements $\xi_i \in \mathcal{A}_{2^i-1}$ and state Milnor's computation of $\mathcal{A}_*$ as a Hopf algebra [Swi75, Theorem 18.20].

(13) **The Adams spectral sequence**
For finite CW spectra X and Y, the $H\mathbb{Z}/p$-based Adams spectral sequence has the form

$$E_2^{s,t} = \mathrm{Ext}^{s,t}_{\mathcal{A}^*}(H^*(Y, \mathbb{Z}/p), H^*(X, \mathbb{Z}/p)) \Rightarrow [\Sigma^t X, Y]^\wedge_p$$

This spectral sequence is one the most powerful tools in stable homotopy theory, and it is used for example to compute stable homotopy groups of spheres (taking $X = Y = \mathbb{S}$). Explain its construction by means of the Adams resolution [Koc96, Proposition 3.6.1], and identify the E_2-term with the Ext-groups over the Steenrod algebra [Koc96, Lemma 3.6.3]. There are many other references for this construction, for example [Rak17, Bru09]. (In [Ada74, §15] and [Swi75, Chapter 19], a more general version of this spectral sequence is constructed, with $H\mathbb{Z}/p$ replaced by an arbitrary ring spectrum.)

(14) **The Pontryagin–Thom construction**
The goal of this talk is to sketch the proof of the Pontryagin–Thom theorem: there is an isomorphism

$$\Omega_* \cong \pi_* \mathrm{MO}$$

between the ring of cobordism classes of closed smooth manifolds and the homotopy ring of MO (and an analogous result for smooth manifolds with a fixed stable tangential structure). A good reference is [Koc96, Theorem 1.5.10].

(15) **The classification of smooth manifolds up to cobordism**
Using the Pontryagin–Thom theorem and the Adams spectral sequence, explain the computation of the cobordism ring Ω_* of unoriented smooth manifolds [Koc96, Theorem 3.7.6], and sketch the computation of the complex cobordism Ω_*^U [Koc96, Theorem 3.7.7].

References

[Ada74] J. F. Adams, *Stable homotopy and generalised homology*, Part III, The University of Chicago Press, 1974

[Boa99] J. M. Boardman, *Conditionally Convergent Spectral Sequences*, 1999, `https://hopf.math.purdue.edu/Boardman/ccspseq.pdf`

[Bru09] R. R. Bruner, *An Adams Spectral Sequence Primer*, 2009, `http://www.rrb.wayne.edu/papers/adams.pdf`

[DP84] A. Dold and D. Puppe, *Duality, trace and transfer*, Proc. Steklov Inst. Math. **154** (1984), pp. 85–103

[HS97] P. J. Hilton and U. Stammbach, *A Course in Homological Algebra*, Springer, 1997

[Koc96] S. O. Kochman, *Bordism, stable homotopy, and Adams spectral sequences*, American Mathematical Society, 1996

[Rak17] A. Raksit, *The Adams spectral sequence*, 2017, `http://stanford.edu/~arpon/files/adams-ss.pdf`

[Swi75] R. M. Switzer, *Algebraic Topology – Homotopy and Homology*, Springer, 1975