REGENSBURG RESEARCH SEMINAR WS 2022/23

ABSOLUTE PRISMATIC COHOMOLOGY

MARC HOYOIS AND DENIS NARDIN

Introduction

Absolute prismatic cohomology is a cohomology theory for p-adic formal schemes developed by Bhatt and Lurie, following the introduction of the prismatic site by Bhatt and Scholze. Roughly speaking, this theory serves as a bridge between the crystalline cohomology of the special fiber and the étale cohomology of the generic fiber. It also leads to a general definition of syntomic cohomology with p-adic coefficients, and ultimately of motivic cohomology with integral coefficients, of arbitrary (derived) schemes.

In this seminar we will learn absolute prismatic cohomology via the (essentially self-contained) article by Bhatt and Lurie [BL22]. Other relevant references are the lecture notes in progress by Bhatt [Bha22] and the work of Drinfeld [Dri22b, Dri22a]. We will in particular review the definitions of prisms and relative prismatic cohomology and learn about the Cartier–Witt stack (also known as the prismatization of $\mathrm{Spf}\,\mathbb{Z}_p$), absolute prismatic cohomology, the Nygaard filtration and the syntomic cohomology of formal schemes and of schemes.

All numbers below refer to [BL22].

Part I: The Cartier–Witt stack

1. **Introduction and background on prisms (26.10)**
   After a brief overview and some motivation, the notions of bounded, perfect, crystalline, and transversal prisms from [BS22] and §2.1 and the standard examples will be reviewed.
2. **Breuil–Kisin twists and the prismatic logarithm (02.11)**
   Introduce the Breuil–Kisin twists of transversal prisms (Construction 2.2.11) and the prismatic logarithm (Proposition 2.3.3). Explain how to use transversal approximation (Proposition 2.4.1) to extend these constructions to general prisms (§2.5). Give the example the q-de Rham prism (Proposition 2.6.1). Explain the variant of the prismatic logarithm defined on the Tate module (§2.6).
3. **The Cartier–Witt stack I (09.11)**
   Define Cartier–Witt divisors and the Cartier–Witt stack WCart (§3.1). Explain the connection with prisms (Construction 3.2.4, Corollary 3.2.10). Describe quasi-coherent sheaves on WCart as prismatic crystals (Proposition 3.3.5). Define the Hodge–Tate ideal sheaf and Breuil–Kisin twists (3.3.8). Explain the description of the Hodge–Tate divisor $\mathrm{WCart}^{\mathrm{HT}}$ as a classifying stack (Theorem 3.4.13).
4. **The Cartier–Witt stack II (16.11)**
   Identify quasi-coherent sheaves on $\mathrm{WCart}^{\mathrm{HT}}$ with complexes equipped with a Sen operator (Theorem 3.5.8, Proposition 3.5.11, Corollary 3.5.13). Define the Frobenius on WCart and explain the relationship with the Hodge–Tate divisor (Proposition 3.6.6, Theorem 3.6.7). Explain how WCart is covered by the q-de Rham prism (Theorem 3.8.3).

Part II: Prismatic cohomology

5. **Review of de Rham complexes and crystalline cohomology (23.11)**
Review the definitions and basic properties of derived Hodge cohomology (Appendix B), of the derived de Rham complex with its Hodge and conjugate filtrations (Appendices D, E), and of crystalline cohomology (Appendix F).
6. **Relative prismatic cohomology (30.11)**
Define prismatic cohomology and the Hodge–Tate complex relative to a bounded prism (4.1.3, 4.1.6, 4.2.1). Explain étale descent (4.1.15), derived descent (4.2.8, 4.2.9) and quasi-syntomic descent (4.3.13). Along the way, review the notion of quasisyntomic morphism (Appendix C).
7. **Absolute prismatic cohomology and Hodge–Tate cohomology (07.12)**
Define the absolute prismatic cohomology of animated rings and of p-adic formal schemes, and the twisted complexes $\mathrm{R}\Gamma^{[m]}_{\Delta}(X)\{n\}$ (§4.4). Mention the comparison with the cohomology of the absolute prismatic site in the quasisyntomic case (Theorem 4.4.30). Define absolute Hodge–Tate complex and its conjugate filtration (§4.5). Prove that the functors $R \mapsto \overline{\Delta}_R\{n\}$ and $R \mapsto \Delta^{[\bullet]}_R\{n\}$ on animated rings preserve sifted colimits (4.5.8, 4.5.10).
8. **Crystalline comparison and the diffracted Hodge complex (14.12)**
Sketch the proof of the comparison with crystalline cohomology (Theorem 4.6.1). Introduce the diffracted Hodge complex with its conjugate filtration and Sen operator (4.7.1). Identify the Hodge–Tate complexes $\overline{\Delta}_R\{n\}$ with the integer-eigenspaces of the diffracted Hodge complex (4.7.5). Explain the comparison between absolute prismatic cohomology and q-de Rham cohomology (Proposition 4.8.8).
9. **The Nygaard filtration (21.12)**
Introduce the Nygaard filtration on the Frobenius-twisted relative prismatic complexes and state its basic properties (§5.1). Compare the Nygaard filtration with the Hodge filtration (Propositions 5.2.3, 5.2.5). Define the absolute Nygaard filtration (5.5.3) and the relative Frobenius (5.7.1, 5.7.5). Describe the associated graded (5.5.8, 5.5.12) and its basic properties (5.5.15, 5.5.18, 5.5.20)
10. **Periodic cyclic homology and global prismatic complexes (11.01)**
Give an overview of Section 6: Review the definition of TP and define the motivic filtration on Tate-filtered $\mathrm{TP}(R)^\wedge_p$, whose associated graded is Nygaard-filtered prismatic cohomology (Theorems 6.2.4, 6.2.8). Explain how this leads to the definition of global prismatic complexes (§6.4).

Part III: Syntomic cohomology

11. **The first Chern class (18.01)**
Define the syntomic complexes $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathfrak{X}, \mathbb{Z}_p(n))$ of a p-adic formal scheme (§7.4) and state its basic properties (7.4.7, 7.4.8, 7.4.11). Prove its vanishing for negative weights (7.4.9). Explain the construction of the first Chern class of a line bundle, using the prismatic logarithm (§7.5). Explain how it refines the crystalline first Chern class, and sketch the proof of Theorem 7.5.6.
12. **Comparison with étale cohomology (25.01)**
Explain how to construct the comparison map from syntomic cohomology to p-adic étale cohomology (Theorem 8.3.1, Lemma 8.3.2). Define the syntomic cohomology of animated rings (§8.4), and show that it is left Kan extended from smooth $\mathbb{Z}$-algebras (Proposition 8.4.10). Describe the weights 0 and 1 (8.4.13, 8.4.14). Explain the statement of Theorem 8.5.1.

13. **Calculations in syntomic cohomology (01.02)**
Sketch the proof of the projective bundle formula for syntomic cohomology (Theorem 9.1.1), which yields higher Chern classes in the usual way (Theorem 9.0.1). Explain the computation of the syntomic cohomology of BGL_n as a polynomial algebra on the Chern classes of the universal bundle, particularly Lemma 9.3.2 for the key case $n = 1$.

References

[Bha22] B. Bhatt, *Prismatic F-gauges*, in progress, 2022, `https://www.math.ias.edu/~bhatt/teaching/mat549f22/lectures.pdf`
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[Dri22a] V. Drinfeld, *A 1-dimensional formal group over the prismatization of* Spf $\mathbf{Z}_p$, 2022, arXiv:2107.11466
[Dri22b] ______, *Prismatization*, 2022, arXiv:2005.04746