WiSe 23/24 Algebraic Topology I
Exercise sheet 1 (due October 27)

Exercise 1.1 (Sequential colimits).

(a) Let
$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \to \cdots$$
be a sequence of topological spaces and continuous maps, let X_∞ be its colimit in Set, and let $i_n\colon X_n \to X_\infty$ be the canonical map. Define a topology on X_∞ as follows: a subset $U \subset X_\infty$ is open if and only if $i_n^{-1}(U)$ is open in X_n for all $n \geq 0$. With this topology, show that X_∞ is a colimit of the sequence $(X_n)_{n\geq 0}$ in the category Top.

(b) The infinite sphere S^∞ is defined as the colimit $\operatorname{colim}_{n\geq 0} S^n$, where S^n is identified with the subspace of S^{n+1} where the last coordinate is zero. Show that S^∞ is contractible.

Exercise 1.2 (Projective spaces). For $n \geq 0$, the *real projective n-space* $\mathbb{RP}^n$ is the quotient of $\mathbb{R}^{n+1} - \{0\}$ by the equivalence relation: $x \sim y$ if and only if there exists $\lambda \in \mathbb{R} - \{0\}$ such that $x = \lambda y$. The equivalence class of a point $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} - \{0\}$ is usually denoted by $[x_0 : \ldots : x_n] \in \mathbb{RP}^n$. The complex projective n-space $\mathbb{CP}^n$ is defined analogously. Prove the following statements:

(a) There is a homeomorphism $\mathbb{RP}^1 \cong S^1$.

(b) There is a homeomorphism $\mathbb{CP}^1 \cong S^2$.

(c) There is a pushout square in Top
$$\begin{array}{ccc} S^1 & \xrightarrow{p} & \mathbb{RP}^1 \\ {\scriptstyle i}\downarrow & & \downarrow \\ D^2 & \longrightarrow & \mathbb{RP}^2, \end{array}$$
where i is the inclusion of the unit circle in the unit disk and $p(x, y) = [x : y]$.

Exercise 1.3 (Products/coproducts in the homotopy category). Show that the canonical functor $\mathsf{Top} \to \mathsf{hTop}$ preserves products and coproducts (indexed by arbitrary sets). In particular, the homotopy category hTop admits products and coproducts.

Exercise 1.4 (Nullhomotopy). A continuous map $f\colon X \to Y$ is called *constant* if there exists $y \in Y$ such that $f(x) = y$ for all $x \in X$. It is called *nullhomotopic* if it is homotopic to a constant map.

Let X be a topological space. Prove that the following assertions are equivalent:

(a) X is contractible, i.e., $X \simeq *$.

(b) The identity map id_X is nullhomotopic.

(c) For every $Y \in \mathsf{Top}$, every continuous map $Y \to X$ is nullhomotopic.

(d) For every $Y \in \mathsf{Top}$, every continuous map $X \to Y$ is nullhomotopic.

(e) X is nonempty and any two self-maps $X \to X$ are homotopic.