WiSe 23/24 Algebraic Topology I
Exercise sheet 10 (due January 12)

Exercise 10.1 (Compactly generated spaces)**.** A topological space X is called *compactly generated* or a *k-space* if it satisfies the following condition: a subset $U \subset X$ is open if and only if, for every compact Hausdorff space K and every continuous map $f\colon K \to X$, $f^{-1}(U)$ is open in K. Let $\mathsf{CGTop} \subset \mathsf{Top}$ denote the full subcategory of compactly generated spaces.

(a) Show that the inclusion functor $i\colon \mathsf{CGTop} \hookrightarrow \mathsf{Top}$ admits a right adjoint $k\colon \mathsf{Top} \to \mathsf{CGTop}$, that is: for every $X \in \mathsf{Top}$ there exist $k(X) \in \mathsf{CGTop}$ and a continuous map $u_X\colon k(X) \to X$ such that the induced map

$$\mathrm{Hom}_{\mathsf{Top}}(Y, k(X)) \to \mathrm{Hom}_{\mathsf{Top}}(Y, X), \quad f \mapsto u_X \circ f,$$

is a bijection for every $Y \in \mathsf{CGTop}$.

(b) Show that every locally compact Hausdorff space is compactly generated.

(c) Show that every first-countable space is compactly generated.

Hint. Use the characterization of closed sets in terms of sequences.

(d) Show that every CW complex is compactly generated.

Hint. Since the inclusion functor i has a right adjoint, it preserves colimits.

Exercise 10.2 (Coverings of CW complexes)**.** Let B be a CW complex and $p\colon E \to B$ a covering map. Show that E has a CW structure with n-skeleton $E^{(n)} = p^{-1}(B^{(n)})$.

Exercise 10.3 (CW structures on projective spaces)**.** Let $n \geq 0$.

(a) Show that $\mathbb{RP}^n$ has a CW structure with

$$(\mathbb{RP}^n)^{(k)} \cong \mathbb{RP}^k$$

for all $k \leq n$.

Hint. This is a generalization of Exercise 1.2(c).

(b) Show that $\mathbb{CP}^n$ has CW structure with

$$(\mathbb{CP}^n)^{(2k)} = (\mathbb{CP}^n)^{(2k+1)} \cong \mathbb{CP}^k$$

for all $k \leq n$.

Exercise 10.4 (Products of CW complexes)**.** Let X and Y be CW complexes. Show that the product $X \times Y$ in the category CGTop admits a CW structure with

$$(X \times Y)^{(n)} = \bigcup_{p+q=n} X^{(p)} \times Y^{(q)}.$$

Remark. With this CW structure, $X \times Y$ is *not* the product of X and Y in the category CW of CW complexes and cellular maps. For example, the diagonal map $X \to X \times X$ is usually not cellular.