WiSe 23/24 Algebraic Topology I
Exercise sheet 11 (due January 19)

Exercise 11.1 (Homology of real projective spaces). Let $n \geq 1$ and let $\mathbb{RP}^n$ be equipped with the CW structure given by $(\mathbb{RP}^n)^{(k)} = \mathbb{RP}^k$ for $k \leq n$ (see Exercise 10.3).

(a) Compute the cellular chain complex of $\mathbb{RP}^n$.

(b) Compute the homology of $\mathbb{RP}^n$ with coefficients in an arbitrary abelian group A.

Exercise 11.2 (Homology of gluings). Compute the homology groups of the following spaces:

(a) The CW complex obtained from S^1 by attaching two 2-cells with attaching maps of degrees 2 and 3.

(b) The space obtained by gluing the boundary of a Möbius band to the circle $\mathbb{RP}^1 \subset \mathbb{RP}^2$.

(c) The 3-manifold obtained by gluing two copies of a solid torus along their boundaries.

Exercise 11.3 (Moore spaces).

(a) Let A be an abelian group and $n \geq 1$. Construct a connected CW complex X such that

$$\tilde{H}_i(X) \cong \begin{cases} A & \text{if } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

(Such a space X is called a *Moore space* for A.)

Hint. Since a subgroup of a free abelian group is free, there exists a short exact sequence

$$0 \to \bigoplus_J \mathbb{Z} \xrightarrow{d} \bigoplus_I \mathbb{Z} \to A \to 0.$$

Construct X so that d is the differential $C^{\mathrm{cell}}_{n+1}(X) \to C^{\mathrm{cell}}_n(X)$.

(b) Let $(A_i)_{i\geq 1}$ be a sequence of abelian groups. Show that there exists a connected CW complex X such that $H_i(X) \cong A_i$ for all $i \geq 1$.

Exercise 11.4 (Euler characteristic of chain complexes). Let C_* be a chain complex of abelian group. The *Euler characteristic* of C_* is defined by

$$\chi(C_*) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{rk} H_i(C_*)$$

where $\operatorname{rk} A = \dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$, provided this is a finite sum of finite terms.

(a) Let

$$0 \to C_* \to D_* \to E_* \to 0$$

be a short exact sequence of chain complexes. Show that $\chi(D_*) = \chi(C_*) + \chi(E_*)$.

(b) Suppose that C_* is a bounded chain complex of finitely generated abelian groups. Show that

$$\chi(C_*) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{rk} C_i.$$

Hint. Proceed by induction on the length of C_*.