WiSe 23/24 Algebraic Topology I
Exercise sheet 12 (due January 26)

Exercise 12.1 (Acyclic spaces). Recall that a topological space X is *acyclic* if $\tilde{H}_*(X) = 0$. There exist acyclic spaces with nontrivial π_1, an example being the open subset of $\mathbb{R}^3$ which is the complement of the Alexander horned ball. In this exercise, we will construct a more concrete example.

Let X be a 2-dimensional CW complex obtained from $S^1 \vee S^1$ be attaching two 2-cells whose attaching maps $S^1 \to S^1 \vee S^1$ are representatives of the elements a^5b^{-3} and $b^3(ab)^{-2}$ in $\pi_1(S^1 \vee S^1) = \langle a, b\rangle$.

(a) By computing cellular chains, show that X is acyclic.

(b) Show that $\pi_1(X)$ is a nontrivial group.

Hint. By Exercise 4.1, we have $\pi_1(X) \cong \langle a, b \mid a^5b^{-3}, b^3(ab)^{-2}\rangle$. To see that the latter group is nontrivial, observe that it acts on a regular dodecahedron, with a (resp. b) a rotation about an axis passing through the center of a face (resp. through a vertex of that face).

Exercise 12.2 (Homology and cohomology of the Klein bottle). Let K be the Klein bottle. Recall that

$$H_i(K, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z} \oplus \mathbb{Z}/2 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Use the universal coefficient theorem to compute

$$H_*(K, \mathbb{Z}/2^r),\ H_*(K, \mathbb{Z}/s),\ H^*(K, \mathbb{Z}),\ H^*(K, \mathbb{Z}/2^r),\ H^*(K, \mathbb{Z}/s)$$

for any $r \geq 1$ and s odd.

(b) Use the Künneth theorem to compute $H_*(K \times K, \mathbb{Z})$.

Exercise 12.3 (The general Borsuk–Ulam theorem).

(a) Let $p\colon E \to B$ be a covering map of degree 2. Show that there is a short exact sequence of chain complexes

$$0 \to C_*(B, \mathbb{Z}/2) \xrightarrow{\tau} C_*(E, \mathbb{Z}/2) \xrightarrow{p_*} C_*(B, \mathbb{Z}/2) \to 0.$$

What goes wrong if you replace $\mathbb{Z}/2$ by $\mathbb{Z}$?

(b) Let $n \geq 1$. A continuous map $f\colon S^n \to S^n$ is called *odd* if $f(-x) = -f(x)$ for all $x \in S^n$. Show that an odd map has odd degree.

Hint. Use (a) for the covering $p\colon S^n \to \mathbb{RP}^n$.

(c) Deduce the general Borsuk–Ulam theorem: for any continuous map $f\colon S^n \to \mathbb{R}^n$, there exists $x \in S^n$ such that $f(x) = f(-x)$ (see Exercise 6.1).

Exercise 12.4 (Chain complexes of free modules)**.** Let R be a principal ideal domain. A chain complex of R-modules C_* is called *free* if each C_n is free.

(a) Let C_* be a free chain complex of R-modules. Show that C_* is a direct sum of complexes that are concentrated in two consecutive degrees and with injective differential.

Hint. Consider the short exact sequences $0 \to Z_n \to C_n \xrightarrow{d} B_{n-1} \to 0$.

(b) Let C_* and D_* be chain complexes of R-modules with C_* free. Show that every map of graded R-modules $H_*(C_*) \to H_*(D_*)$ lifts to a map of chain complexes $C_* \to D_*$. Moreover, show that any two such lifts are chain homotopic.

Hint. By (a), we can reduce to the case where C_* is $C_1 \hookrightarrow C_0$.

(c) Let C_* and D_* be chain complexes of free R-modules with isomorphic homology modules. Show that C_* and D_* are chain homotopy equivalent (i.e., there exists chain maps $f\colon C_* \to D_*$ and $g\colon D_* \to C_*$ that are inverse to one another up to chain homotopy).