WiSe 23/24 Algebraic Topology I
Exercise sheet 13 (due February 2)

Exercise 13.1 (Bockstein homomorphisms)**.** Let m be a positive integer and X a topological space. The short exact sequence of abelian groups

$$0 \to \mathbb{Z}/m \to \mathbb{Z}/m^2 \to \mathbb{Z}/m \to 0$$

induces upon applying $\mathrm{Hom}(C_*(X), -)$ a short exact sequence of chain complexes

$$0 \to C^*(X, \mathbb{Z}/m) \to C^*(X, \mathbb{Z}/m^2) \to C^*(X, \mathbb{Z}/m) \to 0.$$

The *mod m Bockstein homomorphism* $\beta\colon H^i(X, \mathbb{Z}/m) \to H^{i+1}(X, \mathbb{Z}/m)$ is the connecting homomorphism in the associated long exact sequence.

(a) Show that β factors as

$$H^i(X, \mathbb{Z}/m) \xrightarrow{\tilde{\beta}} H^{i+1}(X, \mathbb{Z}) \to H^{i+1}(X, \mathbb{Z}/m),$$

for some map $\tilde{\beta}$. Deduce that $\beta^2 = 0$.

Hint. Consider the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{m} \mathbb{Z} \to \mathbb{Z}/m \to 0,$$

which maps to the one above.

(b) Determine the mod 2 Bockstein homomorphisms on $H^*(\mathbb{RP}^n, \mathbb{Z}/2)$.

Hint. First compute $H^*(\mathbb{RP}^n, \mathbb{Z}/2)$ and $H^*(\mathbb{RP}^n, \mathbb{Z})$.

Exercise 13.2 (Vanishing of cup products)**.** Let X be a topological space and let R be a ring.

(a) Let $A, B \subset X$ be *open* subspaces. Using the Alexander–Whitney map, construct a refined cup product

$$H^p(X, A, R) \otimes H^q(X, B, R) \to H^{p+q}(X, A \cup B, R).$$

Hint. Recall the excision quasi-isomorphism $C_*^{\{A,B\}}(A \cup B) \hookrightarrow C_*(A \cup B)$.

(b) Suppose that X admits an open covering by n contractible subspaces (e.g., $X = \mathbb{RP}^{n-1}$ or $X = \mathbb{CP}^{n-1}$). Show that the product of any n elements of positive degree in $H^*(X, R)$ is zero.

(c) Deduce that the cohomology ring $H^*(\Sigma X, R)$ is a square-zero extension of R.

(d) Compute explicitly the cohomology rings $H^*(S^n)$ for all $n \geq -1$.

Exercise 13.3 (Künneth formula in cohomology). Let R be a principal ideal domain.

(a) Let X and Y be topological spaces such that all homology groups $H_n(X, R)$ and $H_n(Y, R)$ are finitely generated free R-modules (for example, X and Y are finite CW complexes and R is a field). Show that there is a canonical isomorphism

$$H^n(X \times Y, R) \cong \bigoplus_{p+q=n} H^p(X, R) \otimes_R H^q(Y, R).$$

(b) Let $n_1, \dots, n_k \in \mathbb{N}$. Show that there is an isomorphism of graded-commutative rings

$$H^*(S^{n_1} \times \cdots \times S^{n_k}, R) \cong \Lambda_R(x_1, \dots, x_k),$$

where the right-hand side is the exterior algebra with generators x_i in degree n_i.

Hint. Use Exercise 13.1(d) to define a ring homomorphism from the right-hand side to the left-hand side.