WiSe 23/24 Algebraic Topology I
Exercise sheet 2 (due November 3)

Exercise 2.1 (Path-connectedness and homotopy equivalence)**.** Recall that a topological space X is *path-connected* if it is nonempty and every pair of points in X is connected by a path. We say that X is *simply path-connected* if it is path-connected and any two paths in X with the same endpoints are path-homotopic. Prove that these properties depend only on the homotopy type of X, that is:

(a) If X is path-connected and $X \simeq Y$, then Y is path-connected.

(b) If X is simply path-connected and $X \simeq Y$, then Y is simply path-connected.

Remark. In the algebraic topology literature, "simply path-connected" is often called "simply connected".

Exercise 2.2 (Homotopy invariance)**.** Let $F\colon \mathsf{Top} \to \mathsf{C}$ be a functor. Show that the following assertions are equivalent:

(a) F is homotopy invariant, i.e., $F(f) = F(g)$ whenever $f \simeq g$.

(b) F sends homotopy equivalences to isomorphisms.

(c) For every $X \in \mathsf{Top}$, F sends the projection $X \times I \to X$ to an isomorphism.

(d) F factors through the homotopy category hTop, i.e., there exists a functor $\overline{F}\colon \mathsf{hTop} \to \mathsf{C}$ such that the following triangle commutes:

$$\begin{array}{ccc} \mathsf{Top} & \xrightarrow{F} & \mathsf{C}. \\ \downarrow & \nearrow_{\overline{F}} & \\ \mathsf{hTop} & & \end{array}$$

Exercise 2.3 (Actions of groupoids on sets)**.**

(a) Let G be a group. Recall that $\mathsf{B}G$ is a groupoid with one object $*$ whose automorphism group is G (with $g \circ h = gh$). Let Set_G be the category of left G-sets: objects are sets with a left action of G, and morphisms are G-equivariant maps. Show that there is an equivalence of categories

$$\mathrm{Fun}(\mathsf{B}G, \mathsf{Set}) \simeq \mathsf{Set}_G.$$

(b) Let $E\colon \mathsf{C} \to \mathsf{C}'$ be an equivalence of categories. Show that the induced functor

$$\mathrm{Fun}(\mathsf{C}', \mathsf{D}) \to \mathrm{Fun}(\mathsf{C}, \mathsf{D}), \quad F \mapsto F \circ E$$

is an equivalence for any category D.

Remark. Similarly, if $\mathsf{D} \to \mathsf{D}'$ is an equivalence, the induced functor $\mathrm{Fun}(\mathsf{C}, \mathsf{D}) \to \mathrm{Fun}(\mathsf{C}, \mathsf{D}')$ is an equivalence.

(c) Let Γ be a groupoid and O a set of representatives of isomorphism classes of objects in Γ. Using (a) and (b), deduce that there is an equivalence of categories

$$\mathrm{Fun}(\Gamma, \mathsf{Set}) \simeq \prod_{x \in O} \mathsf{Set}_{\mathrm{Aut}_\Gamma(x)}.$$

Exercise 2.4 (Pushouts of groups)**.**

(a) Construct pushouts in the category Grp of groups.

Remark. Grp also admits filtered colimits (unlike pushouts, these are preserved by the forgetful functor $\mathsf{Grp} \to \mathsf{Set}$) and an initial object (the trivial group). It follows that Grp admits all colimits.

The pushout of a diagram of groups

$$\begin{array}{ccc} H & \longrightarrow & G_1 \\ \downarrow & & \\ G_2 & & \end{array}$$

is denoted by $G_1 *_H G_2$ and is also called the *amalgamated product* of G_1 and G_2 over H. When H is the trivial group, we simply write $G_1 * G_2$.

(b) Let G_1 and G_2 be nontrivial groups. Show that $G_1 * G_2$ is infinite and nonabelian.

(c) Let Gpd be the category of groupoids. Show that the functor

$$\mathsf{B} \colon \mathsf{Grp} \to \mathsf{Gpd}$$

(see Exercise 2.3(a)) preserves pushouts.

Hint. Verify the universal property in Gpd.

Remark. The functor B also preserves filtered colimits, but it does not preserve arbitrary colimits, because it does not preserve the initial object. One can fix this by considering B as a functor to the category Gpd_* of *pointed* groupoids.