WiSe 23/24 Algebraic Topology I
Exercise sheet 3 (due November 10)

Exercise 3.1 (π_1 of a filtered union). Let X be a topological space and let $(X_i)_{i\in I}$ be a filtered family of subspaces, i.e., $(I,\leq)$ is a filtered partially ordered set and $X_i \subset X_j$ whenever $i \leq j$. Suppose that X is the union of the interiors of the X_i's:

$$X = \bigcup_i X_i^\circ.$$

(a) Show that $\Pi_1(X) \cong \operatorname{colim}_{i\in I} \Pi_1(X_i)$.

(b) If $x \in \bigcap_{i\in I} X_i$, deduce that $\pi_1(X,x) \cong \operatorname{colim}_{i\in I} \pi_1(X_i,x)$.

Exercise 3.2 (π_1 of a wedge). If $(X_i,x_i)_{i\in I}$ is a collection of pointed spaces, we denote by $\bigvee_{i\in I}(X_i,x_i)$ their coproduct in Top_*, also called their *wedge*, which is explicitly the quotient space

$$\bigvee_{i\in I}(X_i,x_i) = \coprod_{i\in I} X_i \Big/ \coprod_{i\in I}\{x_i\}.$$

A pointed topological space (X,x) is called *well-pointed* if there exists a neighborhood U of x such that the inclusion $U \hookrightarrow X$ is homotopic rel $\{x\}$ to the constant map with value x. Show that the functor $\pi_1\colon \mathsf{Top}_* \to \mathsf{Grp}$ preserves coproducts of well-pointed spaces.

Hint. For finite coproducts, use the Seifert–van Kampen theorem. Deduce the general case by a judicious use of Exercise 3.1(b).

Remark. In the literature, "well-pointed" usually refers to the condition that $X \vee I$ is a retract of $X \times I$, which is stronger than the above.

Exercise 3.3 (π_1 of the projective plane). Use the Seifert–van Kampen theorem to compute $\pi_1(\mathbb{RP}^2,*)$.

Hint. The description of $\mathbb{RP}^2$ from Exercise 1.2 may be useful.

Exercise 3.4 (Higher homotopy groups are abelian). The goal of this exercise is to prove that the homotopy groups π_n are abelian for $n \geq 2$.

(a) Let S be a set equipped with two binary operations $*$ and $\circ$. Suppose that they have a common neutral element $e \in S$ and satisfy the interchange law

$$(a*b)\circ(c*d) = (a\circ c)*(b\circ d).$$

Show that $* = \circ$ and that $a*b = b*a$. This is called the *Eckmann–Hilton argument.*

(b) Let (X,e) be a pointed topological space and $\mu\colon X\times X \to X$ a pointed map such that $\mu(e,-) \simeq_* \mathrm{id}_X \simeq_* \mu(-,e)$. Show that the group $\pi_1(X,e)$ is abelian.

(c) Recall that $\pi_n(X,x)$ is the set of pointed homotopy classes of map $I^n/\partial I^n \to (X,x)$. For each $1 \leq i \leq n$, there is a group operation $*_i$ on $\pi_n(X,x)$ induced by concatenating in the ith direction:

$$(\alpha *_i \beta)(s_1,\dots,s_n) = \begin{cases} \alpha(s_1,\dots,2s_i,\dots,s_n) & \text{if } s_i \in [0,1/2], \\ \beta(s_1,\dots,2s_i-1,\dots,s_n) & \text{if } s_i \in [1/2,1]. \end{cases}$$

If $n \geq 2$, show that all these group operations on $\pi_n(X,x)$ coincide and are abelian.