WiSe 23/24 Algebraic Topology I
Exercise sheet 4 (due November 17)

Exercise 4.1 (π_1 of cell attachments). Let X be a topological space. One says that Y is *obtained from X by attaching n-cells* if there are maps $\phi_i\colon S^{n-1} \to X$ and a pushout square

$$\begin{array}{ccc} \coprod_{i\in I} S^{n-1} & \xrightarrow{(\phi_i)_i} & X \\ \downarrow & & \downarrow \\ \coprod_{i\in I} D^n & \longrightarrow & Y. \end{array}$$

The maps ϕ_i are called the *attaching maps*.

Suppose X path-connected and let $x_0 \in X$.

(a) Suppose that Y is obtained from X by attaching n-cells for some $n \geq 3$. Show that $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ is an isomorphism.

(b) Suppose that Y is obtained from X by attaching 2-cells. For each attaching map $\phi_i\colon S^1 \to X$, choose a path γ_i from x_0 to $\phi_i(1)$, and let $N \subset \pi_1(X, x_0)$ be the normal subgroup generated by the loops $\gamma_i * \phi_i * \bar{\gamma}_i$ for $i \in I$. Show that $\pi_1(Y, x_0) \cong \pi_1(X, x_0)/N$.

(c) Prove that the functor $\pi_1\colon \mathsf{Top}_* \to \mathsf{Grp}$ is essentially surjective.

Hint. Use (b) and Exercise 3.2.

Exercise 4.2 (π_1 of surfaces). Recall that every closed connected surface is homeomorphic to Σ_g for some $g \geq 0$ or to N_h for some $h \geq 1$, where Σ_g (resp. N_h) is obtained from a sphere by attaching g copies of the torus $S^1 \times S^1$ (resp. h copies of the real projective plane $\mathbb{RP}^2$).

For each of the following surfaces, give a presentation of the fundamental group and compute its abelianization as a direct sum of groups of the form $\mathbb{Z}/n$ (recall that the abelianization of a group G is the abelian group $G^{\mathrm{ab}} = G/[G, G]$).

(a) The genus 2 surface Σ_2.

(b) The Klein bottle N_2.

(c) (Optional) The remaining closed surfaces Σ_g and N_h for $g, h \geq 3$.

Exercise 4.3 (The Brouwer fixed-point theorem in low dimensions). The Brouwer fixed-point theorem states that every continuous map $f\colon D^n \to D^n$ has a fixed point. Prove the Brouwer fixed-point theorem for $n = 1, 2$.

Hint. Suppose there exists $f\colon D^n \to D^n$ with no fixed points. For $x \in D^n$, let $r(x)$ be the unique point on S^{n-1} such that $f(x)$, x and $r(x)$ are aligned (in this order). Show that r is a continuous retraction of the inclusion $i\colon S^{n-1} \hookrightarrow D^n$ and derive a contradiction.

Exercise 4.4 (Suspension). If X is a topological space, its *suspension* ΣX is defined by the pushout square

$$\begin{array}{ccc} X \times \{0,1\} & \hookrightarrow & X \times I \\ \downarrow & & \downarrow \\ \{0,1\} & \hookrightarrow & \Sigma X. \end{array}$$

(a) Show that there is a homeomorphism $\Sigma S^n \cong S^{n+1}$ for all $n \geq -1$.

(b) Show that X is path-connected if and only if ΣX is simply path-connected.