WiSe 23/24 Algebraic Topology I

Exercise sheet 5 (due November 24)

Exercise 5.1 (Stability properties of covering maps). Prove the following statements:

(a) Let $p\colon E \to B$ and $p'\colon E' \to B'$ be covering maps. Then $p \times p'\colon E \times E' \to B \times B'$ is a covering map.

(b) Let $p\colon E \to B$ be a covering map and

$$\begin{array}{ccc} E' & \xrightarrow{g} & E \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ B' & \xrightarrow{f} & B \end{array}$$

a pullback square in Top. Then $p'\colon E' \to B'$ is a covering map.

Consider a commutative triangle

$$\begin{array}{ccc} E' & \xrightarrow{f} & E \\ & {\scriptstyle p'}\searrow \quad \swarrow{\scriptstyle p} & \\ & B. & \end{array}$$

(c) Assume that p and p' are covering maps. If B is locally connected or if the fibers of p' are finite, then f is a covering map.

(d) (Optional) Assume that p and f are covering maps. If the fibers of p are finite, then p' is a covering map.

Remark. In general, a composition of covering maps is not a covering map.

(e) (Optional) Assume that p' and f are covering maps with f surjective. If B is locally connected or if the fibers of p' are finite, then p is a covering map.

Exercise 5.2 (Examples of covering maps). Show that the following maps are coverings:

(a) $S^1 \to S^1$, $z \mapsto z^n$ $(n \geq 1)$.

(b) $\mathbb{R} \to S^1$, $r \mapsto e^{ir}$.

(c) $\mathbb{C} \to \mathbb{C} - \{0\}$, $z \mapsto e^z$.

Exercise 5.3 (Connectedness revisited). Let X be a topological space. Consider the functor

$$\tau_X\colon \mathsf{Set} \to \mathsf{Cov}_X, \quad \tau_X(S) = (S \times X \to X),$$

sending a set S to the trivial covering of X with fiber S. Prove the following statements:

(a) X is connected if and only if τ_X is fully faithful.

(b) X is simply connected if and only if τ_X is an equivalence of categories.

Exercise 5.4 (Galois coverings)**.** If $p\colon E \to B$ is a covering, its group of automorphisms $\mathrm{Aut}_B(E)$ in Cov_B is called the group of *deck transformations*. Note that $\mathrm{Aut}_B(E)$ acts on the fiber E_b for each $b \in B$.

Suppose B connected. A covering $p\colon E \to B$ is called *Galois* (or *regular*, or *normal*) if E is connected and $\mathrm{Aut}_B(E)$ acts transitively on each fiber.

(a) Let $p\colon E \to B$ be a Galois covering. Show that $\mathrm{Aut}_B(E)$ acts freely on E and that p induces a homeomorphism $E/\mathrm{Aut}_B(E) \cong B$.

(b) Let G be a (discrete) group acting continuously on a topological space X. Suppose that every $x \in X$ admits a neighborhood U such that $U \cap gU = \emptyset$ for all $g \in G - \{e\}$. Show that the quotient map $\pi\colon X \to X/G$ is a covering. If X is connected, show that π is a Galois covering whose group of deck transformations is isomorphic to G.