WiSe 23/24 Algebraic Topology I
Exercise sheet 6 (due December 1)

Exercise 6.1 (The Borsuk–Ulam theorem in low dimensions)**.** The Borsuk–Ulam theorem states that if $f\colon S^n \to \mathbb{R}^n$ is continuous, then there exists $x \in S^n$ such that $f(x) = f(-x)$. Prove the Borsuk–Ulam theorem for $n = 1, 2$.

Hint. Let $g(x) = f(x) - f(-x)$, and suppose g has no zeros. Then there is a commutative square

$$\begin{array}{ccc} S^2 & \xrightarrow{g/\|g\|} & S^1 \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ \mathbb{RP}^2 & \xrightarrow{\bar{g}} & \mathbb{RP}^1 \end{array}$$

where the vertical maps are coverings. Show that the pullback of p to $\mathbb{RP}^2$ is trivial by computing its monodromy.

Exercise 6.2 (Geometric realization via nondegenerate simplices)**.** If X is a simplicial set, denote by $X_n^{\mathrm{nd}} \subset X_n$ the subset of nondegenerate n-simplices.

(a) Show that, for any simplex x of X, there exists a unique nondegenerate simplex $x^\sharp$ and a unique sequence $i_1 \leq \cdots \leq i_k$ such that $x = s_{i_1} \dots s_{i_k}(x^\sharp)$.

(b) Show that there is a homeomorphism

$$|X| \cong \left(\coprod_{n \geq 0} X_n^{\mathrm{nd}} \times \Delta^n \right) / \sim,$$

where $\sim$ is the equivalence relation generated by $(x, \delta_i(u)) \sim (d_i(x)^\sharp, \sigma_{i_k} \dots \sigma_{i_1}(u))$, where $d_i(x) = s_{i_1} \dots s_{i_k}(d_i(x)^\sharp)$.

Hint. There is an obvious map from the right-hand side to the left-hand side. Use (a) to construct a map in the other direction.

Remark. If the face maps of X preserve nondegenerate simplices, then the latter form a semisimplicial subset X^{nd} and the homeomorphism (b) becomes $|X| \cong \|X^{\mathrm{nd}}\|$.

Exercise 6.3 (Homology of semisimplicial sets)**.** For the following semisimplicial sets X, identify the geometric realization $\|X\|$ and compute the homology groups $H_*(X, \mathbb{Z})$ (if X_n is not specified, it is empty):

(a) $X_0 = \{v\}$, $X_1 = \{e\}$.

(b) $X_0 = \{v\}$, $X_1 = \{e_1, e_2\}$.

(c) $X_0 = \{v_1, v_2\}$, $X_1 = \{a, b, c_1, c_2\}$, $X_2 = \{s, t\}$,
$d_0(a) = d_0(b) = v_2$, $d_1(a) = d_1(b) = v_1$, $d_0(c_1) = d_1(c_1) = v_1$, $d_0(c_2) = d_1(c_2) = v_2$,
$d_{0,1,2}(s) = c_2, b, a$, $d_{0,1,2}(t) = b, a, c_1$

(d) $X_0 = \{v\}$, $X_1 = \{a, b, c\}$, $X_2 = \{s, t\}$,
$d_{0,1,2}(s) = a, c, b$, $d_{0,1,2}(t) = b, c, a$

Exercise 6.4 (Chain homotopy and semisimplicial homotopy)**.** Let C_* and D_* be chain complexes of abelian groups and let $f, g\colon C_* \to D_*$ be chain maps. A *chain homotopy* from f to g is a collection of morphisms $h_n\colon C_n \to D_{n+1}$ such that

$$f_n - g_n = d \circ h_n + h_{n-1} \circ d.$$

(a) Suppose there exists a chain homotopy from f to g. Show that f and g induce the same map $H_*(C_*) \to H_*(D_*)$ on homology.

Let $f, g\colon X \to Y$ be morphisms of semisimplicial objects in some category. A *semisimplicial homotopy* from f to g is a collection of morphisms $h_{n,i}\colon X_n \to Y_{n+1}$ for $0 \leq i \leq n$ such that

$$\begin{aligned} d_0 h_{n,0} &= f_n \\ d_{n+1} h_{n,n} &= g_n \\ d_i h_{n,j} &= \begin{cases} h_{n-1,j-1} d_i & \text{if } i < j, \\ d_i h_{n,j-1} & \text{if } i = j \neq 0, \\ h_{n-1,j} d_{i-1} & \text{if } i > j+1. \end{cases} \end{aligned}$$

(b) Suppose $f, g\colon A \to B$ are morphisms of semisimplicial abelian groups. Show that a semisimplicial homotopy from f to g induces a chain homotopy between the induced chain maps of Moore complexes $f, g\colon C_*(A) \to C_*(B)$.

Hint. Let $h_n = \sum_{i=0}^{n} (-1)^i h_{n,i}$.

(c) Let $f, g\colon X \to Y$ be morphisms of semisimplicial sets and suppose there is a semisimplicial homotopy from f to g. Using (a) and (b), show that f and g induce the same map $H_*(X, A) \to H_*(Y, A)$ on homology with coefficients in any abelian group A.