WiSe 23/24 Algebraic Topology I
Exercise sheet 7 (due December 8)

Exercise 7.1 (π_0 of simplicial sets)**.** If X is a simplicial set, the set $\pi_0(X)$ is defined to be the coequalizer of the pair
$$d_0, d_1 \colon X_1 \rightrightarrows X_0.$$

(a) Show that $\pi_0(X) \cong \pi_0(|X|)$.

(b) For A an abelian group, show that there is an isomorphism $H_0(X, A) \cong \bigoplus_{\pi_0(X)} A$.

(c) Show that the functor $\pi_0 \colon \mathsf{sSet} \to \mathsf{Set}$ is left adjoint to the functor $c \colon \mathsf{Set} \to \mathsf{sSet}$ sending a set S to the constant simplicial set $[n] \mapsto S$; that is, there is a natural transformation $\eta \colon \mathrm{id}_{\mathsf{sSet}} \to c \circ \pi_0$ inducing bijections
$$\mathrm{Hom}_{\mathsf{Set}}(\pi_0(X), S) \cong \mathrm{Hom}_{\mathsf{sSet}}(X, c(S)), \quad f \mapsto c(f) \circ \eta_X.$$

Exercise 7.2 (π_1 and group completion)**.** Let C be a small category. Recall that the *nerve* of C is the simplicial set
$$\mathrm{N}(C) \colon \Delta^{\mathrm{op}} \to \mathsf{Set}, \quad [n] \mapsto \mathrm{Hom}_{\mathsf{Cat}}([n], C).$$

(a) Construct a functor $\varphi_C \colon C \to \Pi_1(|\mathrm{N}(C)|)$ sending objects and morphisms to the corresponding vertices and edges in $|\mathrm{N}(C)|$.

Let M be a monoid and let $\mathsf{B}M$ be the category with a single object $*$, whose endomorphism monoid is M. In this case, $\mathrm{N}(\mathsf{B}M)$ is also called the *bar construction* on M.

(b) Show that $\varphi_{\mathsf{B}M}$ exhibits $\pi_1(|\mathrm{N}(\mathsf{B}M)|, *)$ as the group completion of M; that is, for every group G, every monoid homomorphism $M \to G$ factors uniquely through $\varphi_{\mathsf{B}M} \colon M \to \pi_1(|\mathrm{N}(\mathsf{B}M)|, *)$.

Hint. Compute π_1 using Exercises 6.2, 3.2, and 4.1.

Exercise 7.3 (Homology of simplicial sets)**.** Compute the homology groups of the following simplicial sets ($n \geq 0$):

(a) $\Delta[n]/\partial\Delta[n]$

(b) $\Delta[1]/\partial\Delta[1] \times \Delta[1]/\partial\Delta[1]$

(c) $\Delta[n]$

(d) $\partial\Delta[n]$

Hint. For (c), show that the maps
$$h_{m,i} \colon \Delta[n]_m \to \Delta[n]_{m+1}, \quad h_{m,i}(a_0, \cdots, a_m) = (0, \ldots, 0, a_i, \ldots, a_m),$$
define a semisimplicial homotopy from the identity of $\Delta[n]$ to a constant map and use Exercise 6.4. Deduce (d) from (c).

Exercise 7.4 (Geometric realization and products)**.** Consider the canonical map

$$\alpha_{p,q}\colon |\Delta[p] \times \Delta[q]| \to |\Delta[p]| \times |\Delta[q]| \cong \Delta^p \times \Delta^q.$$

(a) Show that $\alpha_{p,q}$ is a homeomorphism for $p = q = 1$ and for $p = 2, q = 1$. (Draw pictures!)

(b) Assuming the fact that $\alpha_{p,q}$ is a homeomorphism for all p, q, show that the canonical map

$$|X \times Y| \to |X| \times |Y|$$

is a bijection for all simplicial sets X and Y, and a homeomorphism if X has finitely many nondegenerate simplices.

Hint. If X has finitely many nondegenerate simplices, then $|X|$ is locally compact (see Exercise 6.2). The existence of the exponential $(-)^{|X|}$ in Top implies that the functor $(-) \times |X|$ preserves colimits.