WiSe 23/24 Algebraic Topology I
Exercise sheet 8 (due December 15)

**Exercise 8.1** (Abstract Mayer–Vietoris sequence)**.** Consider a commutative diagram of abelian groups

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A_n & \xrightarrow{f_n} & B_n & \xrightarrow{g_n} & C_n & \xrightarrow{\partial_n} & A_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle a_n} & & \downarrow{\scriptstyle b_n} & & \cong\downarrow{\scriptstyle c_n} & & \downarrow{\scriptstyle a_{n-1}} & & \\ \cdots & \longrightarrow & A'_n & \xrightarrow{f'_n} & B'_n & \xrightarrow{g'_n} & C'_n & \xrightarrow{\partial'_n} & A'_{n-1} & \longrightarrow & \cdots \end{array}$$

where the rows are long exact sequences and $c_n$ is an isomorphism for all $n \in \mathbb{Z}$. Verify that the associated "abstract Mayer–Vietoris" sequence

$$\cdots \longrightarrow A_n \xrightarrow{(a_n,-f_n)} A'_n \oplus B_n \xrightarrow{(f'_n,b_n)} B'_n \xrightarrow{\partial_n^{\mathrm{MV}}} A_{n-1} \longrightarrow \cdots$$

is exact, where $\partial_n^{\mathrm{MV}} = \partial_n \circ c_n^{-1} \circ g'_n$.

**Exercise 8.2** (Nerve of natural transformations)**.** Let $C$ and $D$ be small categories, $f, g\colon C \to D$ functors, and $\varphi\colon f \to g$ a natural transformation.

(a) Show that $\varphi$ induces a homotopy between the morphisms of simplicial sets

$$\mathrm{N}(f),\ \mathrm{N}(g)\colon \mathrm{N}(C) \to \mathrm{N}(D)$$

and of spaces

$$|\mathrm{N}(f)|,\ |\mathrm{N}(g)|\colon |\mathrm{N}(C)| \to |\mathrm{N}(D)|.$$

*Hint.* The natural transformation $\varphi$ can be viewed as a functor $C \times [1] \to D$.

(b) Deduce that $|\mathrm{N}(-)|$ takes equivalences of categories to homotopy equivalences of spaces.

(c) Suppose that $C$ has either an initial object or a final object. Show that $|\mathrm{N}(C)|$ is contractible.

(d) If $\Gamma$ is a groupoid, show that the functor $\varphi_\Gamma\colon \Gamma \to \Pi_1(|\mathrm{N}(\Gamma)|)$ from Exercise 7.2(a) is an equivalence.

*Hint.* Use (b) and Exercise 7.2(b).

**Exercise 8.3** (Homology of wedges)**.** Let $(X_\alpha)_{\alpha\in A}$ be a family of topological spaces with base points $x_\alpha \in X_\alpha$. Recall that their coproduct in $\mathsf{Top}_*$ is given by the wedge sum

$$\bigvee_{\alpha\in A} X_\alpha = \coprod_{\alpha\in A} X_\alpha \Big/ \coprod_{\alpha\in A} \{x_\alpha\}.$$

Let $i_\beta\colon X_\beta \hookrightarrow \bigvee_{\alpha\in A} X_\alpha$ be the canonical map and let $p_\beta\colon \bigvee_{\alpha\in A} X_\alpha \twoheadrightarrow X_\beta$ be the pointed map such that $p_\beta \circ i_\alpha$ is the identity on $X_\beta$ if $\alpha = \beta$ and is constant otherwise.

(a) If $A$ is finite, show that the pointed map

$$\bigvee_{\alpha\in A} X_\alpha \to \prod_{\alpha\in A} X_\alpha$$

induced by the maps $p_\beta \circ i_\alpha \colon X_\alpha \to X_\beta$ is a homeomorphism onto its image.

(b) Suppose that, for every $\alpha \in A$, $\{x_\alpha\}$ is closed in $X_\alpha$ and is a neighborhood deformation retract, i.e., there exists a neighborhood $N_\alpha$ of $x_\alpha$ such that $\mathrm{id}_{N_\alpha}$ is homotopic rel $\{x_\alpha\}$ to the constant map. Show that there an isomorphism

$$\tilde{H}_*\left(\bigvee_{\alpha\in A} X_\alpha\right) \cong \bigoplus_{\alpha\in A} \tilde{H}_*(X_\alpha)$$

such that:

- the inclusion $\tilde{H}_*(X_\beta) \hookrightarrow \bigoplus_{\alpha\in A} \tilde{H}_*(X_\alpha)$ is identified with $(i_\beta)_*$,
- the projection $\bigoplus_{\alpha\in A} \tilde{H}_*(X_\alpha) \twoheadrightarrow \tilde{H}_*(X_\beta)$ is identified with $(p_\beta)_*$.

**Exercise 8.4** (Pinch maps)**.** Let $n \geq 1$. We consider the $n$-sphere $S^n \subset \mathbb{R}^{n+1}$ as a pointed topological space with base point $e_1 = (1, 0, \ldots, 0)$. Let

$$i_1, i_2 \colon S^n \hookrightarrow S^n \vee S^n$$

be the summand inclusions and

$$p_1, p_2 \colon S^n \vee S^n \to S^n$$

the projections, as in Exercise 8.3. The *fold map*

$$\mathrm{fold} \colon S^n \vee S^n \to S^n$$

is the unique pointed map such that $\mathrm{fold} \circ i_1 = \mathrm{id}_{S^n}$ and $\mathrm{fold} \circ i_2 = \mathrm{id}_{S^n}$.

(a) Construct a pointed map

$$\mathrm{pinch} \colon S^n \to S^n \vee S^n$$

such that $p_1 \circ \mathrm{pinch} \simeq_* \mathrm{id}_{S^n}$ and $p_2 \circ \mathrm{pinch} \simeq_* \mathrm{id}_{S^n}$.

(b) Let $f, g \colon S^n \to S^n$ be pointed maps. Show that

$$f_* + g_* = (\mathrm{fold} \circ (f \vee g) \circ \mathrm{pinch})_* \colon H_*(S^n) \to H_*(S^n).$$

*Hint.* This is a formal consequence of (a) and Exercise 8.3(b).

*Remark.* The pinch map makes $S^n$ into a cogroup object in the pointed homotopy category $\mathsf{hTop}_*$, which is commutative if $n \geq 2$. This induces the group structure on the homotopy groups of a pointed space.