# WiSe 23/24 Algebraic Topology I
## Exercise sheet 9 (due December 22)

**Exercise 9.1** (Invariance of domain)**.**

(a) Compute $H_*(\mathbb{R}^n, \mathbb{R}^n - 0)$.

(b) Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty opens such that $U \cong V$. Show that $n = m$.

**Exercise 9.2** (Homology of surfaces)**.**

(a) Compute the homology with integral coefficients of the surfaces $\Sigma_g$ and $N_h$.

*Hint.* Use the polygonal presentation of these surfaces and the Mayer–Vietoris sequence.

(b) Compute $H_*(\mathbb{RP}^2, A)$ for every abelian group $A$.

**Exercise 9.3** (Homology of products of spheres)**.** Compute $H_*(S^m \times S^n)$ for all $m, n$.

**Exercise 9.4** (The Hurewicz map)**.** Let $(X, x_0)$ be a pointed topological space and $n \geq 1$. Choose a generator $\sigma \in H_n(S^n, \mathbb{Z}) \cong \mathbb{Z}$. The *Hurewicz map*

$$h_n \colon \pi_n(X, x_0) \to H_n(X, \mathbb{Z})$$

sends a pointed homotopy class $[f \colon (S^n, *) \to (X, x_0)]_*$ to $f_*(\sigma) \in H_n(X, \mathbb{Z})$.

(a) Show that $h_n$ is a group homomorphism.

*Hint.* Express the group structure on $\pi_n(X, x_0)$ using the pinch map on $S^n$; see Exercise 8.4.

(b) If $X$ is path-connected, show that $h_1$ is surjective.

*Remark.* The *Hurewicz theorem* states further that $h_1$ exhibits $H_1(X, \mathbb{Z})$ as the abelianization of $\pi_1(X, x_0)$.