REGENSBURG RESEARCH SEMINAR WS 2023/24

THE $\mathbb{P}^1$-FREUDENTHAL SUSPENSION THEOREM

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The Freudenthal suspension theorem in homotopy theory states that the connectivity of the loop–suspension map $X \to \Omega\Sigma X$ is twice as high as the connectivity of X. In motivic homotopy theory, the algebraic projective line $\mathbb{P}^1$ plays the role of the topological circle S^1, and the existence of a motivic version of the Freudenthal suspension theorem involving $\mathbb{P}^1$ has been a natural open question since the beginning of motivic homotopy theory. It was recently resolved by Asok, Bachmann, and Hopkins. Among other applications, they obtain a proof of Murthy's splitting conjecture on vector bundles of rank just below the dimension. In this seminar we will go through the proofs of the $\mathbb{P}^1$-Freudenthal suspension theorem and of Murthy's conjecture, following [ABH23a].

Section and theorem numbers refer to [ABH23a].

1. **Introduction and overview (24.10)**
2. **The $\mathbb{A}^1$-connectivity theorem (31.10)** [§2.1], [AWW16, §2]
   Introduce strongly and strictly $\mathbb{A}^1$-invariant sheaves and state Morel's theorem [AWW16, Theorem 2.2.7]. Deduce the unstable $\mathbb{A}^1$-connectivity theorem following [AWW16, Theorem 2.2.12] and its immediate consequences [AWW16, §3].
3. **Nilpotent motivic spaces (07.11)** [§2.1], [ABH23b]
   Introduce solvable and nilpotent motivic spaces following [ABH23b]. Prove Propositions 2.1.18 and 2.1.20.
4. **Motivic spectra, t-structures and the slice filtration (14.11)** [§2.2]
   Introduce motivic spectra, the slice filtration, the homotopy t-structure and the effective homotopy t-structure. Review various equivalent descriptions of the hearts of these t-structures (homotopy modules, sheaves with framed/Milnor–Witt transfers) and their completeness properties. In particular, discuss Morel's theorem about $\mathbb{G}_m$-loop spaces (2.1.10), which implies that $\Omega_{\mathbb{G}_m}$ preserves connectivity and sifted colimits of simply connected motivic spaces (Proposition 2.2.5(2)).
5. **Real étale homotopy theory (21.11)** [§2.3], [Bac18]
   Introduce the real étale topology and the corresponding realization of motivic spaces and spectra, following [Bac18]. Explain the relation with ρ-periodization and Proposition 2.3.2. Define the + and − parts of 2-periodic motivic spectra (2.2.22) and prove Theorem 2.3.3.
6. **The homotopy coniveau tower (28.11)** [Lev05]
   Introduce the homotopy coniveau tower of a presheaf of spectra on smooth schemes [Lev05, §2]. Discuss the comparison with the slice filtration [Lev05, Theorems 7.1.1 and 9.0.3]. Deduce Voevodsky's $\mathbb{P}^1$-connectivity conjecture [Lev05, Theorem 7.4.2].
7. **$\mathbb{G}_m$-delooping (05.12)** [§2.2], [BY19, Bac22, Fel20]
   Give an overview of the proof of Theorem 2.2.29.

8. **Weak cellularity and nullification (12.12)** [§3]
Give an overview of §3.1 and §3.2 up to Propositions 3.2.10 and 3.2.11. Discuss also several examples from §3.3.
9. **Refined Whitehead theorem and towers (19.12)** [§4]
Prove the weakly cellular Whitehead theorem 3.2.12 and give an overview of §4. The key results are Amplification 4.1.13, Proposition 4.2.1, Proposition 4.3.3.
10. **The motivic Dold–Thom theorem (09.01)** [§5.1, §5.2], [Voe08]
Review equivariant motivic homotopy theory and the motivic Dold–Thom theorem of Suslin and Voevodsky, following §5.1 and §5.2. Furthermore, explain Voevodsky's comparison theorem between $\mathbb{A}^1$-invariant Nisnevich and cdh sheaves of spectra over a field of characteristic 0 [Voe08].
11. **Cellular estimates for Eilenberg–Mac Lane spaces (16.01)** [§5.3, §5.4]
Explain the proofs of Theorems 5.3.7 and 5.4.2 about the connectivity of the assembly maps of a very effective $H\mathbb{Z}$-module. Time permitting, discuss the case of positive characteristic (§5.5).
12. **The $\mathbb{P}^1$-Freudenthal suspension theorem (23.01)** [§6]
Sketch the proof of Theorem 6.2.1 via Proposition 6.1.2 and Lemmas 6.2.2–6.2.7. Time permitting, discuss Theorem 6.3.1.
13. **Application to algebraic vector bundles (30.01)** [§7.1]
Review the classification theorem for vector bundles in $\mathbb{A}^1$-homotopy theory and explain the proof of Murthy's conjecture in characteristic zero.

References

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