WiSe 25/26 Algebraic Geometry I
Exercise sheet 10 (due January 8)

Exercise 10.1. (2 points) Let k be a ring and $d \in \mathbb{Z}$. Recall that there is a cartesian square of categories

$$\begin{array}{ccc} \mathrm{Mod}_{\mathbb{P}^1_k} & \longrightarrow & \mathrm{Mod}_{k[u]} \\ \downarrow & & \downarrow {\scriptstyle u\mapsto t} \\ \mathrm{Mod}_{k[v]} & \xrightarrow{v\mapsto t^{-1}} & \mathrm{Mod}_{k[t^{\pm 1}]}. \end{array}$$

Under this limit description, show that the line bundle $\mathcal{O}(d)$ on $\mathbb{P}^1_k$ corresponds to the triple $(k[u], k[v], \alpha_d \colon k[u^{\pm 1}] \xrightarrow{\sim} k[v^{\pm 1}])$ where $\alpha_d(u^n) = v^{d-n}$.

Exercise 10.2. (4 points) Let k be a field. In this problem, we compute the category of line bundles over $\mathbb{P}^1_k$. Let $n \in \mathbb{N}$ and let $\mathcal{V}_n(X)$ denote the set of isomorphism classes of vector bundles of constant rank n over X.

(a) Using the description of $\mathrm{Mod}_{\mathbb{P}^1_k}$ recalled in Exercise 10.1, show that there is a bijection

$$\mathcal{V}_n(\mathbb{P}^1_k) \simeq \mathrm{GL}_n(k[t^{\pm 1}])/\sim,$$

where $A \sim B$ if and only if there exists $S \in \mathrm{GL}_n(k[t])$ and $T \in \mathrm{GL}_n(k[t^{-1}])$ such that $B = SAT$.

Hint. As k is a field, $k[t]$ is a principal ideal domain and hence every vector space over $k[t]$ is free.

(b) Show that every line bundle on $\mathbb{P}^1_k$ is isomorphic to $\mathcal{O}(d)$ for a unique $d \in \mathbb{Z}$.

Hint. As k is reduced, $k[t]^\times = k^\times$.

Remark. One can show more generally that, for every $n \geq 1$, every line bundle on $\mathbb{P}^n_k$ is isomorphic to $\mathcal{O}(d)$ for a unique $d \in \mathbb{Z}$.

Remark. Using row and column operations, one can show that any matrix in $\mathrm{GL}_n(k[t^{\pm 1}])$ is $\sim$-equivalent to a diagonal matrix of the form $\mathrm{diag}(t^{d_1}, \ldots, t^{d_n})$ with $d_1, \ldots, d_n \in \mathbb{Z}$. This means that every vector bundle over $\mathbb{P}^1_k$ is isomorphic to a sum of line bundles $\mathcal{O}(d_1) \oplus \cdots \oplus \mathcal{O}(d_n)$. Since we also know that $\mathrm{Map}(\mathcal{O}(a), \mathcal{O}(b)) = k[x, y]_{b-a}$, we know the group of maps between any two such vector bundles, which gives a complete description of the category $\mathrm{Vect}_{\mathbb{P}^1_k}$. Vector bundles over higher-dimensional projective spaces are much more complicated, and there is no general classification of them.

Exercise 10.3. (2 points) Let X be a quasi-affine scheme. Show that the canonical map $X \to \mathrm{Spec}(\mathcal{O}(X))$ is an open immersion.

Hint. By definition, $X \simeq \mathrm{D}(I) \subset \mathrm{Spec}(A)$, where $I \subset A$ is a finite subset. We then know that $\mathcal{O}(X) \simeq \mathrm{L}_I A$. Let $\lambda \colon A \to \mathrm{L}_I A$ be the unit map and consider the cartesian square

$$\begin{array}{ccc} \mathrm{D}(\lambda(I)) & \hookrightarrow & \mathrm{Spec}(\mathrm{L}_I A) \\ \downarrow & & \downarrow {\scriptstyle \mathrm{Spec}(\lambda)} \\ \mathrm{D}(I) & \hookrightarrow & \mathrm{Spec}(A). \end{array}$$

Show that if $\varphi \colon A \to R$ is an element of $\mathrm{D}(I)(R)$, then R is I-local as an A-module, so that φ factors uniquely through λ. Deduce that $\mathrm{D}(\lambda(I)) \to \mathrm{D}(I)$ is an isomorphism and conclude.