WiSe 25/26 Algebraic Geometry I
Exercise sheet 11 (due January 15)

Exercise 11.1. (4 points) Let P be a complete totally ordered set (i.e., a totally ordered set in which every subset admits a supremum).

(a) Show that P is a locale.

(b) Describe the space of points of P, and deduce that P is spatial.

Hint. Recall that points are given by completely prime filters $F \subset P$.

Exercise 11.2. (6 points) Let T be a topological space. Recall that the unit map $\eta_T \colon T \to \mathrm{Pt}(\mathrm{Open}(T))$, which is the initial map from T to a sober space, is given by

$$T \to \{\text{irreducible closed subsets of } T\}, \quad t \mapsto \overline{\{t\}}.$$

(a) Show that η_T is injective if and only if T is a Kolmogorov space (i.e., no two points have exactly the same collection of neighborhoods).

(b) Let k be a field and let A be a finitely generated k-algebra. Show that every radical ideal $I \subset A$ is an intersection of maximal ideals.

Hint. First reduce to the case $A = k[x_1, \dots, x_n]$. Suppose f belongs to every maximal ideal containing I. Let $\bar{k}$ be an algebraic closure of k. For every zero $a = (a_1, \dots, a_n) \in \mathrm{V}(I)(\bar{k})$, the k-subalgebra $k[a_1, \dots, a_n] \subset \bar{k}$ is a field, since the a_i's are algebraic over k. Hence, the polynomials vanishing on a form a maximal ideal of $k[x_1, \dots, x_n]$, which contains I. Conclude that $f \in I$ using Hilbert's Nullstellensatz.

(c) Let k be a field and let A be a finitely generated k-algebra. Let $\mathrm{Max}(A) \subset \mathrm{Prim}(A)$ be the subspace of maximal ideals (also called the *maximal spectrum* of A). Show that the inclusion $\mathrm{Max}(A) \hookrightarrow \mathrm{Prim}(A)$ exhibits $\mathrm{Prim}(A)$ as the soberification of $\mathrm{Max}(A)$.

Hint. Since $\mathrm{Prim}(A)$ is sober, this is equivalent to the statement that $\mathrm{Open}(\mathrm{Prim}(A)) \xrightarrow{\sim} \mathrm{Open}(\mathrm{Max}(A))$. Prove this using (b).

Remark. This gives many examples of Kolmogorov spaces that are not sober. For example, $\mathrm{Max}(k[x])$ is an infinite set equipped with the cofinite topology, which is an irreducible space without a generic point. Its soberification $\mathrm{Prim}(k[x])$ adds the generic point (0), which is the only non-maximal prime ideal in $k[x]$.

Exercise 11.3. (3 points) Let A be an $\mathbb{N}$-graded ring and let $f \in A_d$ be a homogeneous element of degree $d \geq 1$. Show that there is a homeomorphism

$$\{\mathfrak{p} \in \mathrm{hPrim}(A) \mid f \notin \mathfrak{p}\} \xrightarrow{\sim} \mathrm{Prim}(A_{(f)}), \quad \mathfrak{p} \mapsto \mathfrak{p}_{(f)},$$

where the left-hand side is topologized as a subspace of the homogeneous prime spectrum $\mathrm{hPrim}(A)$.