# WiSe 25/26 Algebraic Geometry I
## Exercise sheet 12 (due January 22)

**Exercise 12.1.** (6 points) Let $\varphi\colon R \to S$ be a map of rings and let $X = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(S)$, and $f = \operatorname{Spec}(\varphi)\colon Y \to X$.

(a) Show that, for any $x \in |X|$, the map

$$|Y \times_X \operatorname{Spec}(\kappa(x))| \to |Y| \times_{|X|} \{x\}$$

is a homeomorphism.

Suppose now that $\varphi$ is integral (i.e., that every element of $S$ satisfies a monic polynomial equation over $R$).

(b) If $\varphi$ is also injective, show that $f$ is surjective.

*Hint.* First show that if $S$ is a field, then $R$ is also a field. Deduce that $|f|$ sends maximal ideals to maximal ideals. Using localization at prime ideals, deduce that $|f|$ is surjective.

(c) Deduce from (b) that $f$ is universally closed.

**Exercise 12.2.** (3 points) Let $k$ be any ring, let $C = \operatorname{Spec}(k[x,y]/(y^2 - x^3))$ be the affine cuspidal cubic over $k$, and let $f\colon \mathbb{A}^1_k \to C$ be induced by the $k$-algebra map

$$k[x,y]/(y^2 - x^3) \to k[t], \quad x \mapsto t^2, \quad y \mapsto t^3.$$

Show that $f$ is a universal homeomorphism.

*Hint.* Use Exercise 12.1(a,c).

**Exercise 12.3.** (4 points) Let $k$ be a field, let $N = \operatorname{Spec}(k[x,y]/(y^2 - x^3 - x^2))$ be the affine nodal cubic over $k$, and let $f\colon \mathbb{A}^1_k \to N$ be induced by the $k$-algebra map

$$k[x,y]/(y^2 - x^3 - x^2) \to k[t], \quad x \mapsto t^2 - 1, \quad y \mapsto t(t^2 - 1)$$

(see Exercise 6.1).

(a) Show that $|f|$ identifies $|N|$ with the quotient of $|\mathbb{A}^1_k|$ obtained by identifying the points 1 and $-1$, and hence that $f$ is open.

*Hint.* Compute the fibers of $|f|$ using Exercise 12.1(a), which gives the desired result for the underlying set. Use Exercise 12.1(c) to show that $|N|$ has the quotient topology.

(b) Suppose $\operatorname{char}(k) \neq 2$. Show that $f$ is not universally open.

*Hint.* Consider the pullback of $f$ along itself.

**Exercise 12.4.** (4 points) Let $\mathcal{C}$ be a category and $\tau$ a Grothendieck topology on $\mathcal{C}$. Prove the following statements:

(a) If $R \subset S \subset \text{よ}(X)$ and $R$ is $\tau$-covering, then $S$ is $\tau$-covering.

(b) If $R, S \subset \text{よ}(X)$ are $\tau$-covering sieves, then $R \cap S \subset \text{よ}(X)$ is $\tau$-covering.