WiSe 25/26 Algebraic Geometry I
Exercise sheet 13 (due January 29)

Exercise 13.1. (2 points) Recall that an R-module M is *faithfully flat* if the functor $M \otimes_R (-)\colon \mathrm{Mod}_R \to \mathrm{Mod}_R$ is exact and conservative (i.e., detects isomorphisms). A ring map $\varphi\colon R \to S$ is called *faithfully flat* if S is faithfully flat as an R-module. Show that φ is faithfully flat if and only if it is flat and $\mathrm{Spec}(\varphi)$ is surjective.

Hint. Recall the description of the fibers of $|\mathrm{Spec}(\varphi)|$ from Exercise 12.1(a).

Exercise 13.2. (2 points) Let $\mathcal{C}$ be a small category and let $\mathcal{E} \subset \mathrm{P}(\mathcal{C})$ be a full subcategory admitting a left exact left adjoint $L\colon \mathrm{P}(\mathcal{C}) \to \mathcal{E}$. Let τ be the collection of sieves $R \subset \text{よ}(X)$ such that $L(R) \xrightarrow{\sim} L(\text{よ}(X))$. Show that τ is a topology on $\mathcal{C}$.

Exercise 13.3. (8 points) In this exercise, we prove the "comparison lemma" for sheaves. Let $\mathcal{C}$ be a small category and $\mathcal{D} \subset \mathcal{C}$ a full subcategory. Recall that the inclusion functor $u\colon \mathcal{D} \hookrightarrow \mathcal{C}$ induces adjunctions

$$\mathrm{P}(\mathcal{C}) \begin{array}{c} \xleftarrow{\;u_!\;} \\ \xrightarrow{\;u^*\;} \\ \xleftarrow{\;u_*\;} \end{array} \mathrm{P}(\mathcal{D}),$$

where $u^*(F) = F|_{\mathcal{D}}$ and u_* and $u_!$ are fully faithful.

Let τ be a topology on $\mathcal{C}$ and suppose that $\mathcal{D}$ is *dense* in $(\mathcal{C}, \tau)$. Let $\tau|_{\mathcal{D}}$ be the collection of sieves of the form $u^*(R) \subset \text{よ}_{\mathcal{D}}(X)$ with $X \in \mathcal{D}$ and $R \subset \text{よ}_{\mathcal{C}}(X)$ τ-covering. Prove the following statements:

(a) $\tau|_{\mathcal{D}}$ is a topology on $\mathcal{D}$.

(b) u_* sends $\tau|_{\mathcal{D}}$-sheaves to τ-sheaves.

Hint. Since u_* is right adjoint to u^*, this is equivalent to: for every τ-covering sieve $R \subset \text{よ}_{\mathcal{C}}(X)$, the $\tau|_{\mathcal{D}}$-sheafification of $u^*(R) \hookrightarrow u^*(\text{よ}_{\mathcal{C}}(X))$ is an isomorphism. To prove this, note that $u^*(R) \hookrightarrow u^*(\text{よ}_{\mathcal{C}}(X))$ is a colimit of $\tau|_{\mathcal{D}}$-covering sieves.

(c) u^* sends τ-sheaves to $\tau|_{\mathcal{D}}$-sheaves.

Hint. By adjunction as in (b), this is equivalent to: for every $\tau|_{\mathcal{D}}$-covering sieve $u^*(R) \subset \text{よ}_{\mathcal{D}}(X)$, $u_!(u^*(R)) \to u_!(\text{よ}_{\mathcal{D}}(X)) = \text{よ}_{\mathcal{C}}(X)$ is a τ-local isomorphism. Use density to show: if f in $\mathrm{P}(\mathcal{C})$ is such that $u^*(f)$ is a $\tau|_{\mathcal{D}}$-local epimorphism, then f is a τ-local epimorphism. Deduce that both $u_!(u^*(R)) \to \text{よ}_{\mathcal{C}}(X)$ and its diagonal are τ-local epimorphisms.

(d) The adjunction (u^*, u_*) restricts to an equivalence $\mathrm{Shv}_\tau(\mathcal{C}) \simeq \mathrm{Shv}_{\tau|_{\mathcal{D}}}(\mathcal{D})$.

Hint. By (b) and (c), (u^*, u_*) restricts to an adjunction between categories of sheaves. Since u_* is fully faithful, it remains to show that $u^*\colon \mathrm{Shv}_\tau(\mathcal{C}) \to \mathrm{Shv}_{\tau|_{\mathcal{D}}}(\mathcal{D})$ is conservative, i.e., that if $f\colon F \to G$ is a map of τ-sheaves such that $u^*(f)$ is an isomorphism, then f is an isomorphism. Given $X \in \mathcal{C}$, choose a τ-covering $(Y_i \to X)_{i \in I}$ with $Y_i \in \mathcal{D}$ and write the associated equalizer diagrams for F and G. Deduce that f is a monomorphism, and hence an isomorphism.

Exercise 13.4. (4 points) Let T be a topological space. Given a presheaf $F \in \mathrm{P}(\mathrm{Open}(T))$ and a point $x \in T$, the *stalk* of F at x is the set

$$F_x = \operatorname*{colim}_{x \in U} F(U),$$

where the colimit is indexed by the poset of open neighborhoods of x.

(a) Show that a map $f\colon F \to G$ in $\mathrm{P}(\mathrm{Open}(T))$ is a local epimorphism (with respect to the canonical topology on $\mathrm{Open}(T)$) if and only if, for each $x \in T$, the induced map $f_x\colon F_x \to G_x$ is surjective.

(b) Deduce analogous charaterizations of local monomorphisms and local isomorphisms.