# WiSe 25/26 Algebraic Geometry I
## Exercise sheet 14 (due February 5)

**Exercise 14.1.** (3 points) Let $T$ be a topological space and $S$ a set. Let $\mathrm{Const}_S$ be the constant presheaf on $\mathrm{Open}(T)$ with value $S$. Show that the sheafification of $\mathrm{Const}_S$ is the sheaf $\mathrm{LConst}_S$ of locally constant $S$-valued functions.

*Hint.* First, define the presheaf $\mathrm{LConst}_S$ and show that it is a sheaf on $T$. Then, show that the map $\mathrm{Const}_S \to \mathrm{LConst}_S$ is a local isomorphism, so that it becomes an isomorphism after sheafifying.

**Exercise 14.2.** (6 points) An algebraic functor $X$ is called *separated* if its diagonal $\Delta_X \colon X \to X \times X$ is a closed immersion. Prove the following statements:

(a) Affine schemes are separated.

(b) If $A$ is an $\mathbb{N}$-graded ring, then $\mathrm{Proj}(A)$ is separated.

*Hint.* By Exercise 8.2, it suffices to show that the base change of the diagonal to each affine open subscheme $\mathrm{D}(f) \times \mathrm{D}(g)$ is a closed immersion.

(c) If $X$ is separated, any subfunctor $Y \subset X$ is separated.

(d) If $X$ is separated and

$$Z \to Y \rightrightarrows X$$

is an equalizer diagram, then $Z \to Y$ is a closed immersion.

**Exercise 14.3.** (4 points)

(a) Let $X$ be a scheme. Show that the diagonal map $\Delta_X \colon X \to X \times X$ is an immersion.

*Hint.* Use the analogue of Exercise 8.2 for immersions, and Exercise 14.2(a,c).

(b) Construct a scheme $X$ for which $\Delta_X$ is a non-closed open immersion.

*Hint.* Let $X$ be the gluing of two copies of $\mathrm{Spec}(\mathbb{Z})$ along an open subscheme $U$. To compute $X \times X$, use the fact that, in any category of sheaves, products preserve colimits in each variable (since this holds in Set).

**Exercise 14.4.** (5 points) Let $X$ be a Zariski sheaf on $\mathrm{CAlg}^{\mathrm{op}}$ and let $A, B \subset X$ be subsheaves that are disjoint, i.e., $A \cap B = \varnothing$. Prove the following statements:

(a) The map $A \sqcup B \to X$ is a monomorphism (where $\sqcup$ is the coproduct of sheaves).

(b) If $A$ and $B$ are closed, then $A \sqcup B \hookrightarrow X$ is a closed immersion.

(c) If $A$ and $B$ are open, then $A \sqcup B \hookrightarrow X$ is an open immersion.

*Hint.* Statements (b) and (c) reduce by definition to the case $X$ affine.