WiSe 25/26 Algebraic Geometry I
Exercise sheet 2 (due October 30)

Exercise 2.1. (7 points) Construct the left adjoints to the following forgetful functors (k is any ring):

(a) $\mathrm{Mod}_k \to \mathrm{Set}$

(b) $\mathrm{Mod}_k \to \mathrm{Ab}$

(c) $\mathrm{CAlg}_k \to \mathrm{CAlg}$

(d) $\mathrm{CAlg}_k \to \mathrm{Mod}_k$

(e) $\mathrm{CAlg}_k \to \mathrm{Ab}$

(f) $\mathrm{CAlg}_k \to \mathrm{Set}$

(g) $\mathrm{CAlg}_k \to \mathrm{CMon}$, sending A to the multiplicative monoid $(A, \cdot)$.

Hint. Recall that if R and R' have left adjoints L and L', then $R \circ R'$ has left adjoint $L' \circ L$.

Exercise 2.2. (2 points) Let $f\colon Y \to X$ be a map of sets and let $f_\sharp\colon \mathrm{Set}_{/Y} \to \mathrm{Set}_{/X}$ be the forgetful functor. Describe explicitly the right adjoint f^* to $f_\sharp$ and the right adjoint f_* to f^*.

Hint. To construct f_*, consider first the case $X = *$.

Exercise 2.3. (4 points) Consider the following systems of polynomial equations over $\mathbb{Z}$:

$$\Sigma_1\colon \begin{cases} x^2 + y^2 + 1 \\ x^2 - y \end{cases} \qquad \Sigma_2\colon \begin{cases} x^4 + x^2 + 1 \\ x - y \end{cases} \qquad \Sigma_3\colon \begin{cases} x^4 + x^2 + 1 \\ x^2 - xy \end{cases}$$

Determine which of their solution functors coincide as subfunctors of $\mathbb{A}^2$ and which are isomorphic as affine schemes. If the subfunctors are different, find an explicit finitely presented $\mathbb{Z}$-algebra A such that the solutions in A differ.