WiSe 25/26 Algebraic Geometry I
Exercise sheet 3 (due November 6)

Exercise 3.1. (6 points) Let $k \to k'$ be a ring map. Recall that the *Weil restriction* functor

$$\mathrm{R}_{k'/k} \colon \mathrm{Fun}(\mathrm{CAlg}_{k'}, \mathrm{Set}) \to \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$$

is defined by $\mathrm{R}_{k'/k}(X)(R) = X(R \otimes_k k')$.

Let $d \in \mathbb{N}$ and suppose that k' is free of rank d as a k-module (e.g., $k \to k'$ is a field extension of degree d).

(a) Show that $\mathrm{R}_{k'/k}(\mathbb{A}^1_{k'}) \simeq \mathbb{A}^d_k$.

(b) Deduce that $\mathrm{R}_{k'/k}$ sends affine k'-schemes to affine k-schemes.

Hint. Any affine scheme is the kernel of a map between affine spaces, and Weil restriction preserves limits.

(c) Compute the ring A such that $\mathrm{Spec}(A) \simeq \mathrm{R}_{\mathbb{Z}[\zeta_3]/\mathbb{Z}}(\mathrm{SL}_2)$.

Exercise 3.2. (4 points) Let X be an algebraic functor. Prove the following statements:

(a) For any family $(F_i)_{i\in I}$ of subsets of $\mathcal{O}(X)$, we have

$$\bigcap_{i\in I} \mathrm{V}(F_i) = \mathrm{V}\big(\bigcup_{i\in I} F_i\big),$$
$$\bigcup_{i\in I} \mathrm{D}(F_i) \subset \mathrm{D}\big(\bigcup_{i\in I} F_i\big).$$

Moreover, the inclusion is an equality on local rings.

(b) For subsets $F_1, \ldots, F_n \subset \mathcal{O}(X)$, we have

$$\mathrm{D}(F_1) \cap \cdots \cap \mathrm{D}(F_n) = \mathrm{D}(F_1 \cdots F_n),$$
$$\mathrm{V}(F_1) \cup \cdots \cup \mathrm{V}(F_n) \subset \mathrm{V}(F_1 \cdots F_n),$$

where $F_1 \cdots F_n = \{f_1 \cdots f_n \mid f_i \in F_i\}$. Moreover, the inclusion is an equality on integral domains.

Exercise 3.3. (2 points) Let $0 \leq r \leq n$. Show that there exists an open subfunctor $\mathrm{Mat}_n^r \subset \mathrm{Mat}_n$ such that, for any field k, $\mathrm{Mat}_n^r(k)$ is the set of $n \times n$ matrices of rank $\geq r$.

Hint. A matrix over a field has rank $\geq r$ if and only if it contains an invertible $r \times r$ submatrix.

Exercise 3.4. (4 points) Consider the algebraic functor

$$X \colon \mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \{(f, e) \mid f \in R,\ e \in R/(f), \text{ and } e^2 = e\},$$

known as the *affine line with doubled origin.* Towards a justification of this name:

(a) Construct two subfunctors $U_0, U_1 \subset X$, each isomorphic to $\mathbb{A}^1$, such that $U_0 \cap U_1$ is isomorphic to $\mathbb{A}^1 - 0$.

(b) Show that U_i is open in X.

Hint. An idempotent element is equal to 1 if and only if it is a unit.