WiSe 25/26 Algebraic Geometry I
Exercise sheet 4 (due November 13)

Exercise 4.1. (2 points) Show that closed immersions and open immersions are stable under base change, i.e., for any cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X, \end{array}$$

if f is a closed or open immersion, so is f'.

Exercise 4.2. (5 points) Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be maps of algebraic functors. Prove the following statements:

(a) If f and g are closed immersions, so is $f \circ g$. If $f \circ g$ is a closed immersion and f is a monomorphism, then g is a closed immersion.

Hint. For the second statement, factor g as $Z \to Z \times_X Y \to Y$ and use Exercise 4.1.

(b) If f and g are open immersions, so is $f \circ g$. If $f \circ g$ is an open immersion and f is a monomorphism, then g is an open immersion.

Hint. For the first statement, suppose U is open in $\mathrm{D}(F) \subset \mathrm{Spec}(A)$. For each $f \in F$, there exists $H_f \subset A$ such that $\mathrm{D}(fH_f) = U \cap \mathrm{D}(f)$. Show that $U = \mathrm{D}(H)$ where $H = \bigcup_{f \in F} fH_f$.

Exercise 4.3. (3 points) A subfunctor $Y \subset X$ is called *locally closed* is there exists an open subfunctor $U \subset X$ containing Y as a closed subfunctor. A map of algebraic functors is called an *immersion* if it is a monomorphism whose image is locally closed.

Show that the composition of two immersions is an immersion.

Hint. It suffices to show that if $Z \subset X$ is closed and $U \subset Z$ is open, then $U \subset X$ is locally closed. First prove this when X is affine.

Exercise 4.4. (4 points) The *closure* of a subfunctor $Y \subset X$ is the smallest closed subfunctor $\bar{Y} \subset X$ containing Y (which exists as intersections of closed subfunctors are closed). If $X = \mathrm{Spec}(A)$ and Y is closed in $\mathrm{D}(F)$ for some *finite* set $F \subset A$, show that $Y = \bar{Y} \cap \mathrm{D}(F)$. Deduce that closed subfunctors of quasi-affine schemes are quasi-affine.

Hint. For each $f \in F$, let $J(f) \subset A_f$ be the ideal of $Y \cap \mathrm{D}(f) \subset \mathrm{D}(f)$, and let $I(f)$ be its preimage in A, so that $I(f)_f = J(f)$. Show that $\bar{Y} = \mathrm{V}\big(\bigcap_{f \in F} I(f)\big)$ and that $J(g) \subset I(f)_g$ for all $f, g \in F$. Since F is finite and localization is exact, $\big(\bigcap_{f \in F} I(f)\big)_g = J(g)$.

Remark. If F is not finite, then $Y \subset \bar{Y}$ need not be open. This contrasts with the topological analogue of this situation: if X is a topological space and $Y \subset X$ is a locally closed subset (i.e., Y is closed in some open subset of X), then Y is always open in its closure.