WiSe 25/26 Algebraic Geometry I
Exercise sheet 5 (due November 20)

Exercise 5.1. (3 points) Recall that a k-algebra A is *of finite type* (resp. *of finite presentation*) if it is isomorphic to $k[\Sigma]$ where Σ is a system of polynomial equations over k with finitely many variables (resp. finitely many variables and equations). Prove the following statements:

(a) A is of finite type if and only if, for every k-algebra R which is a filtered union of subalgebras $(R_i)_{i\in I}$,
$$\mathrm{Spec}(A)(R) = \bigcup_{i\in I} \mathrm{Spec}(A)(R_i).$$

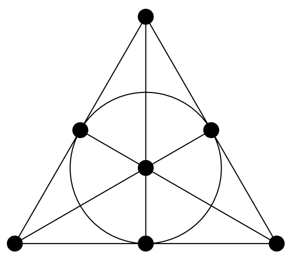
(b) A is of finite presentation if and only if $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ preserves filtered colimits.

Exercise 5.2. (4 points) Let k be a field and $-1 \leq d \leq n$. A *d-dimensional subspace* of $\mathbb{P}^n(k)$ is a subset $S \subset \mathbb{P}^n(k)$ whose preimage in $k^{n+1} - \{0\}$ has the form $V - \{0\}$ for some $(d+1)$-dimensional subspace $V \subset k^{n+1}$. Note that 0-dimensional subspaces of $\mathbb{P}^n(k)$ are just points. Subspaces of $\mathbb{P}^n(k)$ of dimension 1, 2, and $n-1$ are also called *lines*, *planes*, and *hyperplanes*.

(a) Using the standard symmetric bilinear form on k^{n+1}, construct an involution
$$(-)^*\colon \{\text{subspaces of } \mathbb{P}^n(k)\} \xrightarrow{\sim} \{\text{subspaces of } \mathbb{P}^n(k)\},$$
which reverses the incidence relation (i.e., $S \subset T$ if and only if $T^* \subset S^*$) and exchanges d-dimensional subspaces and $(n-1-d)$-dimensional subspaces (e.g., points and hyperplanes). This is called *projective duality*.

Remark. In the familiar case $k = \mathbb{R}$ (and more generally when k is an ordered field), a subspace $S \subset \mathbb{P}^n(k)$ and its dual S^* are always disjoint. But this is not the case in general: for example, there are points $x \in \mathbb{P}^1(\mathbb{C})$ such that $\{x\}^* = \{x\}$.

(b) The following picture is called the *Fano plane*:



Explain how the Fano plane is a representation of points and lines in $\mathbb{P}^2(\mathbb{F}_2)$, and describe explicitly the duality between them.

Exercise 5.3. (3 points) Let R be a ring and let L be a line over R.

(a) Show that the canonical map $L^{\otimes d} \to \mathrm{Sym}^d_R(L)$ from the dth tensor power (over R) to the dth symmetric power of L is an isomorphism.

Choose an R-module L' with an isomorphism $L \otimes_R L' \simeq R$. We then define the negative tensor powers of L by $L^{\otimes(-d)} = (L')^{\otimes d}$ for any $d > 0$.

(b) Show that $\bigoplus_{d\in\mathbb{Z}} L^{\otimes d}$ has a structure of R-algebra with the following universal property: it is initial among R-algebras A with an R-linear map $L \to A$ whose image generates the unit ideal.

Exercise 5.4. (3 points) Let $R_1, \dots, R_n$ be rings and let $R = R_1 \times \cdots \times R_n$.

(a) Show that the projections $R \to R_i$ induce an equivalence of categories

$$\mathrm{Mod}_R \simeq \mathrm{Mod}_{R_1} \times \cdots \times \mathrm{Mod}_{R_n},$$

which restricts to equivalences

$$\mathrm{Vect}_R \simeq \mathrm{Vect}_{R_1} \times \cdots \times \mathrm{Vect}_{R_n},$$
$$\mathrm{Line}_R \simeq \mathrm{Line}_{R_1} \times \cdots \times \mathrm{Line}_{R_n}.$$

Hint. Apply Zariski descent with the "unit vectors" $e_i \in R$, noting that $R_{e_i} \simeq R_i$ and $R_{e_i e_j} = 0$ if $i \neq j$.

(b) Deduce that, for any set I, $\mathbb{P}^I(R) \simeq \mathbb{P}^I(R_1) \times \cdots \times \mathbb{P}^I(R_n)$.