WiSe 25/26 Algebraic Geometry I
Exercise sheet 6 (due November 27)

Exercise 6.1. (5 points) Let k be a ring and let $R = k[x,y]/(y^2 - x^3 - x^2)$ be the ring of functions on the affine nodal cubic over k.

(a) Show that there is an injective ring map

$$R \hookrightarrow k[t], \quad x \mapsto t^2 - 1, \quad y \mapsto t(t^2-1),$$

which induces an isomorphism $R[x^{-1}] \xrightarrow{\sim} k[t,(t^2-1)^{-1}]$.

(b) Let L be the R-submodule of $k[t]$ generated by t and $t^2 - 1$. Show that L is a line over R, which is nontrivial if $k \neq 0$.

Hint. Show that L becomes a trivial line after localizing at x and at $x+1$; for the latter, note that there are well-defined R-linear maps $R \to L$ and $L \to R$ given by $f \mapsto tf$. To see the nontriviality of L, consider the degree of a would-be generator.

Remark. When $k = \mathbb{R}$, this line is an algebro-geometric refinement of the Möbius band (which is the unique nontrivial line bundle over a circle).

Exercise 6.2. (3 points) Let $R = \mathbb{Z}[\sqrt{-5}]$. Show that the ideal $I = (3, 1+\sqrt{-5}) \subset R$ is a nontrivial line over R.

Hint. Show that I becomes principal in $R[\frac{1}{2}]$, and deduce that it is a line. To see that I is not principal, suppose $I = (u)$, so that $3 = uv$ for some $v \in R$. Establish the following facts to derive a contradiction: there is no $x \in R$ with $|x|^2 = 3$; $I \neq R$; $I \neq (3)$.

Exercise 6.3. (2 points) Let R be a ring, $r \in \mathbb{N}$, and V a vector space over R. Show that the locus where V has constant rank r is both closed and open in $\operatorname{Spec}(R)$.

Exercise 6.4. (6 points) Let k be a ring and let $n \geq 0$. The goal of this problem is to prove that every function on $\mathbb{P}^n_k$ is constant.

(a) Let R be a ring, L a line over R, and $(s_i)_{i \in I}$ a generating family in L. Show that, for every R-module M, there is an equalizer diagram

$$M \to \prod_{i \in I} M_{s_i} \rightrightarrows \prod_{i,j \in I} M_{s_i s_j},$$

where $s_i s_j \in L^{\otimes 2}$.

Hint. Choose a generating set $F \subset L^\vee$ and for any $s\colon R \to L$ let $F_s = s^\vee(F) \subset R$. Note that F_s generates the unit ideal in R_s and that $F_s F_t$ generates the unit ideal in R_{st}, and use Zariski descent to write each M_{s_i} and $M_{s_i s_j}$ as an equalizer. Then note that $\bigcup_{i \in I} F_{s_i}$ generates the unit ideal in R, and use Zariski descent once more.

(b) Let $U_i = \mathrm{D}(x_i) \subset \mathbb{P}^n_k$, so that $U_i \simeq \mathbb{A}_k^{\{0,\ldots,\hat{\imath},\ldots,n\}}$. Let $(f_0, \ldots, f_n)$ be a family of functions $f_i \in \mathcal{O}(U_i)$ such that f_i and f_j agree on $U_i \cap U_j$. Show that there exists $a \in k$ such that each f_i is constant with value a, i.e., factors as

$$U_i \to \operatorname{Spec}(k) \xrightarrow{a} \mathbb{A}^1.$$

Hint. Note that $U_i \cap U_j = \mathrm{D}(x_j)$ as a subfunctor of $U_i \simeq \operatorname{Spec}(k[x_0, \ldots, \widehat{x_i}, \ldots, x_n])$.

(c) Deduce that $\mathcal{O}(\mathbb{P}^n_k) \simeq k$.

Hint. Let $f \in \mathcal{O}(\mathbb{P}^n_k)$. By (b), there exists $a \in k$ such that the restriction of f to each U_i is constant with value a. For any $x\colon \operatorname{Spec}(R) \to \mathbb{P}^n_k$, use (a) to deduce that $f \circ x$ is constant with value a. Conclude that f is constant.