WiSe 25/26 Algebraic Geometry I
Exercise sheet 7 (due December 4)

Exercise 7.1. (2 points) Let X be any vanishing locus in $\mathbb{P}^n$. Show that the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ induces a bijection $X(\mathbb{Z}) \xrightarrow{\sim} X(\mathbb{Q})$.

Remark. This result fails if X is an infinite-dimensional projective space.

Exercise 7.2. (4 points) Let k be $\mathbb{Z}[\frac{1}{2}]$-algebra and let $X_2 = \mathrm{V}(x^2 - y^2 - z^2) \subset \mathbb{P}^2_k$ be the degree 2 Fermat curve over k.

(a) Show that X_2 is isomorphic to $\mathbb{P}^1_k$ via

$$\mathbb{P}^1_k \xrightarrow{\sim} X_2,$$
$$[a : b] \mapsto [a^2 + b^2 : a^2 - b^2 : 2ab],$$

where $[a : b]$ is any quotient line $(a, b)\colon R^2 \twoheadrightarrow L$.

Hint. This is not easy to prove by hand. Realize this map as Proj of a degree 2 map between graded rings and use that $\mathrm{Proj}(A) \simeq \mathrm{Proj}(A^{(2)})$.

(b) Find all Pythagorean triples, i.e., all $(a, b, c) \in \mathbb{N}^3$ with $a^2 = b^2 + c^2$.

Hint. Use (a) on $\mathbb{Q}$-points combined with Exercise 7.1.

Exercise 7.3. (4 points) Let A be a $\mathbb{Z}$-graded ring.

(a) Show that there is an action of the multiplicative group $\mathbb{G}_\mathrm{m}$ on $\mathrm{Spec}(A)$ given by

$$(\lambda\varphi)(a) = \lambda^d \varphi(a)$$

for any $\varphi\colon A \to R$, $\lambda \in R^\times$, $d \in \mathbb{Z}$, and $a \in A_d$. Moreover, show that this action restricts to $\mathrm{V}(H)$ and $\mathrm{D}(H)$ for any homogeneous ideal $H \subset A$.

Suppose now that A is an $\mathbb{N}$-graded ring generated by $A_{\leq 1}$. Let $A_+ \subset A$ be the irrelevant ideal and $\mathrm{D}(A_+) \subset \mathrm{Spec}(A)$ its nonvanishing locus.

(b) Show that the map

$$\mathrm{D}(A_+) \to \mathrm{Proj}(A), \quad (\varphi\colon A \to R) \mapsto (\varphi_1\colon A_1 \otimes_{A_0} R \twoheadrightarrow R),$$

is $\mathbb{G}_\mathrm{m}$-invariant and induces a monomorphism

$$\mathrm{D}(A_+)/\mathbb{G}_\mathrm{m} \hookrightarrow \mathrm{Proj}(A),$$

whose image is exactly the subfunctor of trivial quotient lines.

Exercise 7.4. (6 points) In Exercise 7.3(a), we showed that a $\mathbb{Z}$-grading on a ring A induces an action of $\mathbb{G}_\mathrm{m}$ on $\mathrm{Spec}(A)$.

(a) For any ring A, show that this construction defines a bijection

$$\{\mathbb{Z}\text{-gradings on } A\} \xrightarrow{\sim} \{\mathbb{G}_\mathrm{m}\text{-actions on } \mathrm{Spec}(A)\}.$$

Hint. Under the equivalence $\mathrm{CAlg}^{\mathrm{op}} \simeq \mathrm{Aff}$, a $\mathbb{G}_\mathrm{m}$-action on $\mathrm{Spec}(A)$ is a map of rings $A \to A[u^{\pm 1}]$ satisfying certain properties. Let A_d be the preimage of Au^d.

(b) Upgrade this bijection to an equivalence of categories

$$(\mathrm{CAlg}^{\mathbb{Z}})^{\mathrm{op}} \simeq \mathbb{G}_{\mathrm{m}}\mathrm{Aff}$$

between $\mathbb{Z}$-graded rings and affine schemes with $\mathbb{G}_{\mathrm{m}}$-action.

Hint. The construction of Exercise 7.3(a) defines a functor $(\mathrm{CAlg}^{\mathbb{Z}})^{\mathrm{op}} \to \mathbb{G}_{\mathrm{m}}\mathrm{Aff}$.

(c) Consider $\mathbb{A}^1$ as a monoid under multiplication, so that $\mathbb{G}_{\mathrm{m}} \subset \mathbb{A}^1$ is a submonoid. Show that the bijection of (a) restricts to a bijection

$$\{\mathbb{N}\text{-gradings on } A\} \simeq \{\mathbb{A}^1\text{-actions on } \mathrm{Spec}(A)\},$$

and deduce an equivalence of categories

$$(\mathrm{CAlg}^{\mathbb{N}})^{\mathrm{op}} \simeq \mathbb{A}^1\mathrm{Aff}$$

between $\mathbb{N}$-graded rings and affine schemes with $\mathbb{A}^1$-action.