WiSe 25/26 Algebraic Geometry I
Exercise sheet 8 (due December 11)

Exercise 8.1. (3 points) Let k be a ring and let A be a finite k-algebra. Show that $\mathrm{Spec}(A)$ is a projective k-scheme.

Hint. Write $A = k[x_1, \ldots, x_n]/I$ and let $a_i \in A$ be the image of x_i. Since A is finite over k, each a_i satisfies a monic polynomial equation over k. Form the projective closure $B = k[x_0, x_1, \ldots, x_n]/I^{\mathrm{h}}$ and let $\alpha\colon B \to A[t]$ be the graded map sending x_0 to t and x_i to $a_i t$. Show that $\mathrm{Proj}(\alpha)$ is defined and is an isomorphism.

Exercise 8.2. (6 points) Let $X\colon \mathrm{CAlg} \to \mathrm{Set}$ satisfy Zariski descent, i.e., for any ring R and family $(f_i)_{i\in I}$ generating the unit ideal, the diagram

$$X(R) \to \prod_{i\in I} X(R_{f_i}) \rightrightarrows \prod_{i,j\in I} X(R_{f_i f_j})$$

is an equalizer. Let $u\colon X \to \mathrm{Spec}(A)$ be a map, and let $(f_i)_{i\in I}$ generate the unit ideal in A. Suppose that each map $u_i\colon X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A_{f_i}) \to \mathrm{Spec}(A_{f_i})$ is a closed immersion.

(a) Show that u is a monomorphism.

Hint. This only uses the injectivity part of the equalizer diagram for X.

(b) Let $K_i \subset A_{f_i}$ be the ideal of u_i. Show that there is a unique ideal $K \subset A$ such that $K_{f_i} = K_i$ for all i.

Hint. By Zariski descent for modules, the functor $R \mapsto \{\text{ideals in } R\}$ satisfies Zariski descent.

(c) Show that u is a closed immersion.

Hint. Using the assumption on X, construct a comparison map $\mathrm{Spec}(A/K) \to X$ over $\mathrm{Spec}(A)$ and show that it is surjective.

Remark. The same result holds for open immersions and for immersions, with variations of this proof.

Exercise 8.3. (2 points) Let $M \twoheadrightarrow N$ be a surjective k-linear map and let $n \in \mathbb{N}$. Show that the induced map $\mathrm{Gr}_n(N) \to \mathrm{Gr}_n(M)$ is a closed immersion.

Exercise 8.4. (3 points) Prove the projective Nullstellensatz: If k is an algebraically closed field and $n \in \mathbb{N}$, the construction $I \mapsto \mathrm{V}(I)(k)$ defines a bijection

$$\{\text{saturated radical homogeneous ideals in } k[x_0, \ldots, x_n]\} \xrightarrow{\sim} \{\text{algebraic subsets of } \mathbb{P}^n(k)\}.$$

Hint. For surjectivity, show that $\sqrt{I}^{\mathrm{sat}}$ is a radical ideal such that $\mathrm{V}(I)(k) = \mathrm{V}(\sqrt{I}^{\mathrm{sat}})(k)$ (this only uses that k is a reduced ring). For injectivity, consider the intersections with each $U_i = \mathbb{A}^{\{0,\ldots,\hat{\imath},\ldots,n\}}(k)$ and apply the affine Nullstellensatz.