WiSe 25/26 Algebraic Geometry I
Exercise sheet 9 (due December 18)

Exercise 9.1. (7 points) The goal of this problem is to show that clopen subfunctors (i.e., subfunctors that are simultaneously closed and open) are classified by idempotent functions.

(a) Let R be a ring and $I \subset R$ an ideal such that $\mathrm{V}(I)$ is open in $\mathrm{Spec}(R)$. Show that the canonical map $\varphi\colon R \to R/I \times R/\Gamma_I R$ is an isomorphism, where $\Gamma_I R \subset R$ is the ideal of I-nilpotent elements.

Hint. Let J be the radical ideal such that $\mathrm{V}(I) = \mathrm{D}(J)$. The condition $\mathrm{V}(I) \subset \mathrm{D}(J)$ is equivalent to $I + J = R$, while the condition $\mathrm{D}(J) \subset \mathrm{V}(I)$ is equivalent to $I \subset \Gamma_J R$ (use the computation of $\mathcal{O}(\mathrm{D}(J))$). By the first condition, the map $R \to \mathcal{O}(\mathrm{D}(J)) \times \mathcal{O}(\mathrm{D}(I))$ is injective, hence φ is injective. For surjectivity, first show that I is finitely generated using the first condition. Then, using the second condition, show that $J \subset \sqrt{\Gamma_I R}$ and conclude by the Chinese remainder theorem.

(b) Deduce that there is a bijection

$$\mathrm{Idem}(R) \xrightarrow{\sim} \mathrm{Clopen}(\mathrm{Spec}(R)), \quad e \mapsto \mathrm{V}(e),$$

where $\mathrm{Idem}(R)$ is the set of idempotent elements of R and $\mathrm{Clopen}(X)$ is the set of clopen subfunctors of X.

(c) Generalize the bijection of (b) to an arbitrary algebraic functor X:

$$\mathrm{Idem}(\mathcal{O}(X)) \xrightarrow{\sim} \mathrm{Clopen}(X).$$

Exercise 9.2. (6 points)

(a) Construct a line bundle $\mathcal{O}(1)$ on $\mathbb{P}^n$ such that, for any R-point $x\colon \mathrm{Spec}(R) \to \mathbb{P}^n$ classifying a quotient line $R^{n+1} \twoheadrightarrow L$, $\mathcal{O}(1)(x) = L$.

Since $\mathcal{O}(1)$ is a line bundle, we obtain line bundles $\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$ for all $d \in \mathbb{Z}$.

(b) For all $d \in \mathbb{N}$, construct a map

$$s\colon \mathbb{Z}[x_0, \dots, x_n]_d \to \Gamma(\mathbb{P}^n, \mathcal{O}(d)),$$

such that the value of the section $s(f)$ on a quotient line $a\colon R^{n+1} \twoheadrightarrow L$ is $f(a) \in L^{\otimes d}$.

Remark. We will see later that the map s is an isomorphism.

(c) On $\mathbb{P}^1$, show that there is a short exact sequence of vector bundles

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^2 \to \mathcal{O}(1) \to 0.$$

Hint. This boils down to the following statement in linear algebra: if $R^2 \twoheadrightarrow L$ is a quotient line, its kernel is isomorphic to $L^\vee$ (naturally in R).

Exercise 9.3. (4 points) For a family $(x_i)_{i\in I}$ in a poset, we denote by $\bigvee_{i\in I} x_i$ its supremum and by $\bigwedge_{i\in I} x_i$ its infimum.

(a) Let R be a ring. Show that the following distributivity law holds in the poset Rad_R of radical ideals in R: for any $K \in \mathrm{Rad}_R$ and any family $(L_i)_{i\in I}$ in Rad_R,

$$K \wedge \bigvee_{i\in I} L_i = \bigvee_{i\in I} (K \wedge L_i).$$

Hint. Use that $\sqrt{KL} = \sqrt{K \cap L}$. Note that this distributivity law does *not* hold in the poset Id_R of all ideals.

(b) Deduce the same property for the poset Rad_X of quasi-coherent radical ideals over any algebraic functor X.

Hint. First consider how base change $\mathrm{Rad}_R \to \mathrm{Rad}_S$ interacts with $\vee$ and $\wedge$.