ALGEBRAIC GEOMETRY

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Contents

Conventions and notation.

- Throughout, "ring" means "commutative ring" and "algebra" means "commutative algebra". Not necessarily commutative algebras will be called "associative algebras".
- We denote by CAlg the category of (commutative) rings. Given $R \in \mathrm{CAlg}$, we denote by CAlg_R the category of (commutative) R-algebras.
- Given a ring R and a subset $S \subset R$, we denote by $R[S^{-1}]$ the localization of R at S. When $S = \{f\}$ has single element, we also write R_f or $R[\frac{1}{f}]$ for the localization.
- The words "morphism" and "map" are used interchangeably. We denote by $\mathrm{Map}_{\mathcal{C}}(X, Y)$ the set of maps from X to Y in a category $\mathcal{C}$; we simply write $\mathrm{Map}(X, Y)$ if $\mathcal{C}$ is clear from the context.
- The symbols $\simeq$ and $\xrightarrow{\sim}$ are used for isomorphisms within a category as well as for equivalences of categories. The arrows $\hookrightarrow$ and $\twoheadrightarrow$ are sometimes used for monomorphsims and epimorphisms.
- Given a category $\mathcal{C}$ and an object $X \in \mathcal{C}$, we denote by $\mathcal{C}_{/X}$ the category of objects over X and by $\mathcal{C}_{X/}$ the category of objects under X. For example, $\mathrm{CAlg}_R \simeq \mathrm{CAlg}_{R/}$.
- $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is the category of functors from $\mathcal{C}$ to $\mathcal{D}$ (objects are functors, morphisms are natural transformations).

1. Some examples

1.1. **Punctured elliptic curves.** Consider the polynomial equation in two variables $y^2 = x^3 - x$. We can consider its set of solutions in any ring R, namely

$$X(R) = \{(a, b) \in R^2 \mid b^2 = a^3 - a\}.$$

This defines a functor $X\colon \mathrm{CAlg} \to \mathrm{Set}$ from the category of rings to the category of sets, which is an example of an *affine scheme*. Being defined by a single equation in two variables, X is called a *family*

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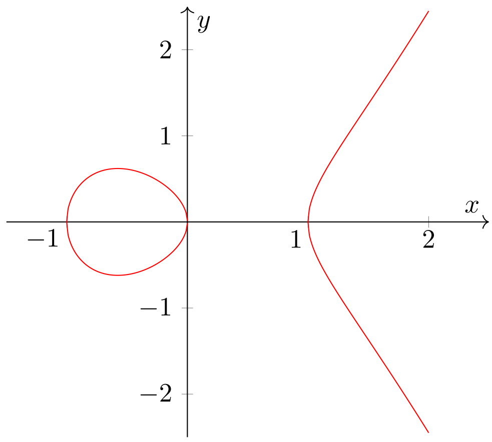
of algebraic curves or simply a *curve*. It is also called an *arithmetic surface*: "arithmetic" because the equation has integral coefficients, and "surface" because it turns out to be a 2-dimensional object from the perspective of dimension theory in commutative algebra.

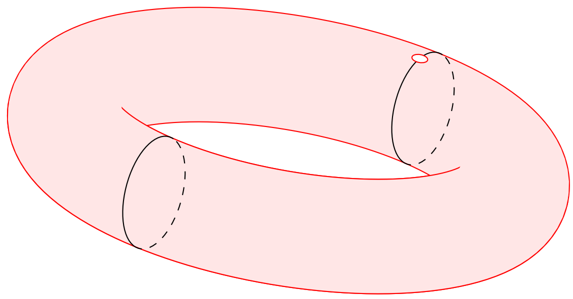
Let us consider explicitly the solution sets $X(R)$ for various rings R:

(i) $X(\mathbb{R})$ is a 1-dimensional real submanifold of $\mathbb{R}^2$, which is diffeomorphic to $S^1 \sqcup \mathbb{R}$:



(ii) For any subring $R \subset \mathbb{R}$, $X(R)$ is the subset of $X(\mathbb{R})$ consisting of all points whose coordinates lie in R. One can show for example that the only points of $X(\mathbb{R})$ with rational coordinates are those on the x-axis, so that $X(\mathbb{Q}) = X(\mathbb{Z}) = \{(0,0), (\pm 1, 0)\}$.

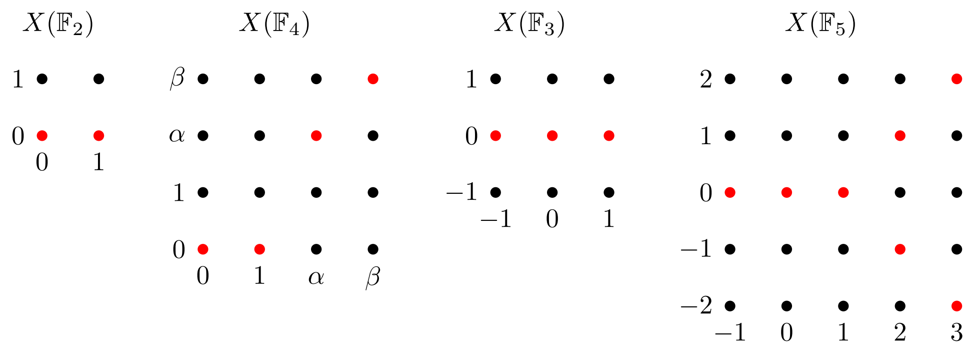
(iii) $X(\mathbb{C})$ is a 1-dimensional complex submanifold of $\mathbb{C}^2$, which can be shown to be diffeomorphic to a punctured torus:



The two black circles indicate the intersection of $X(\mathbb{C})$ with $\mathbb{R}^2 \subset \mathbb{C}^2$, which is the set $X(\mathbb{R})$ from (i). Furthermore, $X(\mathbb{C})$ is biholomorphic to $\mathbb{C}/\Lambda - \{0\}$ where Λ is the lattice $\mathbb{Z} \oplus \mathbb{Z}i$ (and $\mathbb{C}/\Lambda$ means the quotient of abelian groups). This is an example of a noncompact *Riemann surface.*

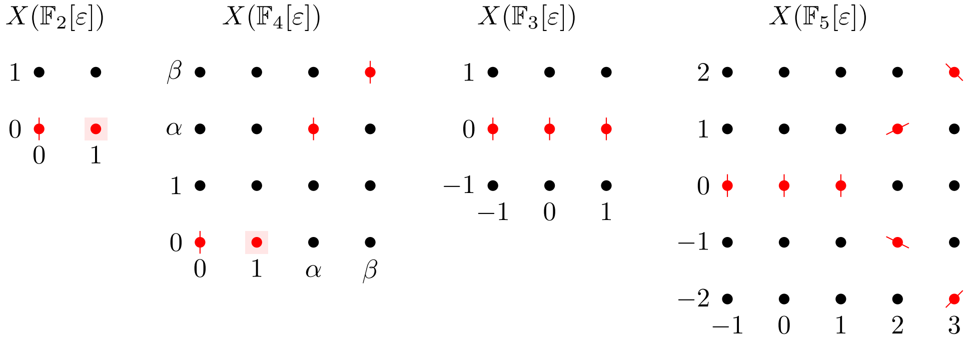
(iv) Let $\mathbb{R}[\varepsilon] = \mathbb{R}[x]/(x^2)$ be the ring of dual numbers over $\mathbb{R}$. A pair of dual numbers $(a + u\varepsilon, b + v\varepsilon)$ belongs to $X(\mathbb{R}[\varepsilon])$ if and only if $(a,b) \in X(\mathbb{R})$ and (u,v) is a tangent vector to $X(\mathbb{R})$ at the point (a,b) (see Remark 1.1 below). Thus, $X(\mathbb{R}[\varepsilon])$ is naturally identified with the *tangent bundle* of the 1-dimensional manifold $X(\mathbb{R})$. As a subspace of $\mathbb{R}[\varepsilon]^2 \simeq \mathbb{R}^4$, $X(\mathbb{R}[\varepsilon])$ is diffeomorphic to $(S^1 \times \mathbb{R}) \sqcup (\mathbb{R} \times \mathbb{R})$.

(v) We can consider solutions in finite fields:



(vi) In analogy with (iv), we can interpret $X(R[\varepsilon])$ as the set of tangent vectors to $X(R)$ for any ring R. Given $(a,b) \in X(R)$, the set of all (u,v) such that $(a + u\varepsilon, b + v\varepsilon)$ belongs to $X(R[\varepsilon])$ is an

R-submodule of R^2, called the *tangent space* of X at (a,b). The tangent spaces over the first few finite fields are as follows:



For any field k, the tangent spaces at all points of $X(k)$ are 1-dimensional k-vector spaces, except for the point $(1,0)$ in characteristic 2, where the tangent space is 2-dimensional. This reflects the fact that the point $(1,0)$ is *singular* in characteristic 2, as it is a meeting point of the two branches through $(1,0)$ and $(-1,0)$. The equation $y^2 = x^3 - x$ is said to have *bad reduction* at the prime 2, and *good reduction* at all other primes.

As the picture (iii) over the complex numbers strongly suggests, one should be able to *compactify* X by filling in the puncture. In the picture (i) over the real numbers, this corresponds to adding a point "at infinity" in the vertical direction that closes up the right-hand component. We will explain how to make sense of this compactified object in §1.3.

Remark 1.1 (Dual numbers and tangent vectors)**.** Let us give some details on the claim in (iv). Given a ring R, a polynomial $f \in R[x_1, \dots, x_n]$, and an n-tuple of dual numbers $a + u\varepsilon \in R[\varepsilon]^n$, we have

$$f(a + u\varepsilon) = f(a) + \left(\frac{\partial f}{\partial x_1}(a)u_1 + \cdots + \frac{\partial f}{\partial x_n}(a)u_n\right)\varepsilon$$

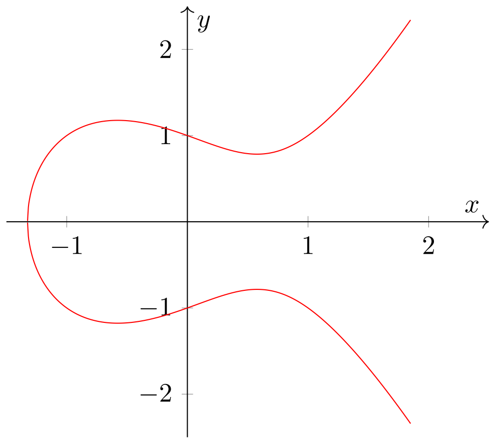
in $R[\varepsilon]$. This expression vanishes if and only if a is a zero of f and u is orthogonal to the gradient $\nabla f(a)$. If $R = \mathbb{R}$ and if this gradient is not zero, so that $f^{-1}(0)$ is a submanifold of $\mathbb{R}^n$ in a neighborhood of a, this precisely means that u is tangent to $f^{-1}(0)$ at a. On the other hand, if this gradient is zero, then every vector is orthogonal to it. Analogous statements hold for $R = \mathbb{C}$.

Consider now the slightly different equation $y^2 = x^3 - x + 1$ and associated solution sets

$$Y(R) = \{(a,b) \in R^2 \mid b^2 = a^3 - a + 1\}.$$

Despite the similarity to the previous equation $y^2 = x^3 - x$ defining X, Y turns out to be qualitatively quite different from X:

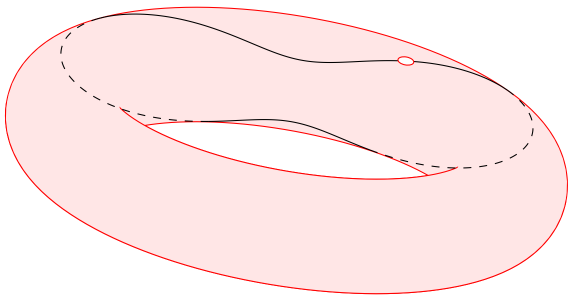
(vii) The set $Y(\mathbb{R}) \subset \mathbb{R}^2$ now has a single component and is diffeomorphic to $\mathbb{R}$:



(viii) The set $Y(\mathbb{Q})$ of rational solutions is infinite, and there are exactly 12 solutions in $\mathbb{Z}$:

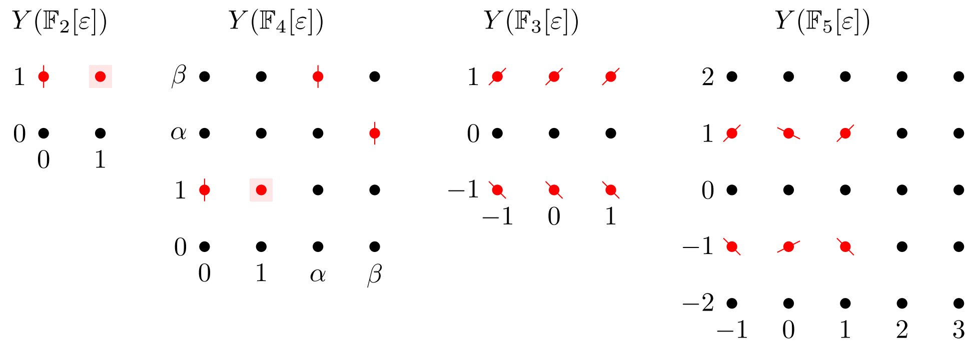
$$Y(\mathbb{Z}) = \{(-1,\pm 1), (0,\pm 1), (1,\pm 1), (3,\pm 5), (5,\pm 11), (56,\pm 419)\}.$$

(ix) The set $Y(\mathbb{C}) \subset \mathbb{C}^2$ is again a punctured torus, but its intersection with $\mathbb{R}^2$ now has a single component:



It is biholomorphic to $\mathbb{C}/\Lambda - \{0\}$ for some lattice $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$ where $\tau \approx \frac{1}{2} + 0.233i$. An exact expression for τ can be written using certain integrals called *elliptic integrals*, which also appear in the formula for the arc length of an ellipse; this is the origin of the term "elliptic curve". It turns out that $X(\mathbb{C})$ and $Y(\mathbb{C})$ are *not* biholomorphic, even though they are diffeomorphic.

(x) Here are the solutions over some finite fields together with their tangent spaces:

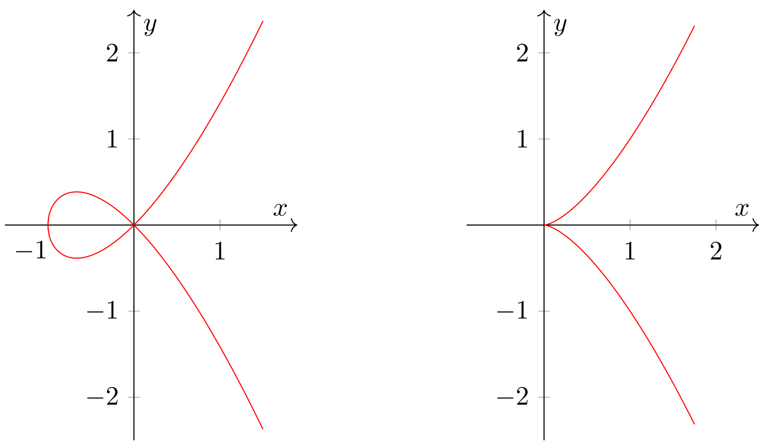


In addition to the prime 2 shown above, the equation $y^2 = x^3 - x + 1$ also has bad reduction at the prime 23, as $(0, 13)$ is a singular point of $Y(\mathbb{F}_{23})$. It has good reduction at all other primes.

1.2. **Singular cubic curves.** Consider the polynomial equations $y^2 = x^3 + x^2$ and $y^2 = x^3$. As in §1.1, they define functors $N, C\colon \mathrm{CAlg} \to \mathrm{Set}$ given by

$$N(R) = \{(a, b) \in R^2 \mid b^2 = a^3 + a^2\},$$
$$C(R) = \{(a, b) \in R^2 \mid b^2 = a^3\}.$$

We call N the *nodal cubic* and C the *cuspidal cubic*. Here is what $N(\mathbb{R})$ and $C(\mathbb{R})$ look like:



In contrast to the curves in §1.1, these curves have singularities and hence do not define submanifolds of $\mathbb{R}^2$. We can "confirm" the singular nature of the node and of the cusp by computing tangent spaces as explained in Remark 1.1. A vector (u, v) belongs to the tangent space at $(a, b) \in N(R)$ if and only if

$$2bv = (3a^2 + 2a)u.$$

For the point $(a, b) = (0, 0)$, this holds for all (u, v), so that the tangent space at $(0, 0)$ is 2-dimensional. Similarly, the equation for the tangent space at $(a, b) \in C(R)$ is

$$2bv = 3a^2u,$$

which always holds if $(a, b) = (0, 0)$. These equations also show that, for any field k, $(0, 0)$ is the only singular point in both $N(k)$ and $C(k)$.

Over the complex numbers, the nodal cubic $N(\mathbb{C}) \subset \mathbb{C}^2$ looks like a punctured *pinched torus*, i.e., a torus in which one of the circles bounding a hole is collapsed to a point; this is equivalently a punctured sphere in which two points have been identified. The cuspidal cubic $C(\mathbb{C}) \subset \mathbb{C}^2$ looks like a punctured sphere with a single thorn.

The two ways of visualizing $N(\mathbb{C})$ correspond to two ways to algebraically "resolve" the nodal singularity of N at the origin:

(i) The "pinched torus" picture suggests viewing N as a degenerate member of a *family* of curves N_λ given (for example) by the equations $y^2 = x(x+\lambda)(x+1)$. We have $N_0 = N$ and, for nonzero values of λ, $N_\lambda(\mathbb{C})$ is a punctured torus with no singular points, similar to both examples in §1.1. Thus, we can think of N as the limit of the nonsingular curves N_λ as $\lambda \to 0$. In this situation, we say that N_λ with $\lambda \neq 0$ is a *deformation* of N.

(ii) The "sphere with two points identified" picture suggests viewing N as a *quotient* of a nonsingular curve $\tilde{N}$ such that $\tilde{N}(\mathbb{C})$ is a punctured sphere. One can achieve this algebraically by a change of variable: if we replace the coordinate y by the "slope" coordinate $s = y/x$, the equation $y^2 = x^3 + x^2$ becomes $s^2 = x + 1$. If $\tilde{N}\colon \mathrm{CAlg} \to \mathrm{Set}$ is the functor defined by the latter equation, i.e.,

$$\tilde{N}(R) = \{(a, c) \in R^2 \mid c^2 = a + 1\},$$

then there is map $\tilde{N} \to N$ sending (a, c) to (a, ac), and one can check that $\tilde{N}$ has no singular points. The curve $\tilde{N}$ is called the *blowup* of N at the origin. Over the real numbers, $\tilde{N}(\mathbb{R}) \to N(\mathbb{R})$ looks like the quotient map $\mathbb{R} \twoheadrightarrow \mathbb{R}/((-1) \sim 1)$.

While an arbitrary system of polynomial equations does not in general have a nonsingular deformation as in (i), it is a deep theorem that, if we work over a field of characteristic zero, it is always possible to resolve singularities by blowing up as in (ii). Whether this is always possible in positive characteristic is a major open question in algebraic geometry.

1.3. **Compactifying affine curves.** The curves considered in §1.1 and §1.2 can be compactified by replacing the ambient 2-dimensional affine space $\mathbb{A}^2$ by the 2-dimensional projective space $\mathbb{P}^2$.

The *affine n-space* $\mathbb{A}^n$ is the functor $\mathrm{CAlg} \to \mathrm{Set}$ given by $\mathbb{A}^n(R) = R^n$. The *projective n-space* $\mathbb{P}^n$ is a functor $\mathrm{CAlg} \to \mathrm{Set}$ containing $\mathbb{A}^n$; for simplicity, we shall only define it here on a certain subcategory of CAlg (the general definition will be given in §3). Let $(\mathbb{A}^n - 0)(R) \subset R^n$ be the set of n-tuples that generate the unit ideal of R. The group of units $R^\times$ acts on the set R^n by scalar multiplication, and this action preserves the subset $(\mathbb{A}^n - 0)(R)$. If R is a local ring or a principal ideal domain, we define

$$\mathbb{P}^n(R) = (\mathbb{A}^{n+1} - 0)(R)/R^\times$$

(for arbitrary rings R, the right-hand side is only a subset of the left-hand side). We write

$$[a_0 : \dots : a_n] \in \mathbb{P}^n(R)$$

for the equivalence class of $(a_0, \dots, a_n) \in (\mathbb{A}^{n+1} - 0)(R)$. We can identify $\mathbb{A}^n(R) = R^n$ with the subset of $\mathbb{P}^n(R)$ where a_0 is a unit:

$$\mathbb{A}^n(R) \hookrightarrow \mathbb{P}^n(R), \quad (a_1, \dots, a_n) \mapsto [1 : a_1 : \dots : a_n].$$

If k is a field, then $\mathbb{P}^n(k)$ is obtained from $\mathbb{A}^n(k)$ by adding a "point at infinity" $[0 : a_1 : \dots : a_n]$ in the direction of every nonzero vector $(a_1, \dots, a_n)$; these points at infinity form a copy of $\mathbb{P}^{n-1}(k)$. Inductively, we therefore obtain a decomposition

$$\mathbb{P}^n(k) = k^n \sqcup k^{n-1} \sqcup \dots \sqcup k^0.$$

A polynomial is called *homogeneous of degree d* if it is a linear combination of monomials of degree exactly d. For example, $x^3 + x^2y - 2xz^2 \in \mathbb{Z}[x, y, z]$ is homogeneous of degree 3. If $f \in R[x_0, \dots, x_n]$ is homogeneous of degree d and $(a_0, \dots, a_n) \in R^{n+1}$, then

$$f(\lambda a_0, \dots, \lambda a_n) = \lambda^d f(a_0, \dots, a_n)$$

for any $\lambda \in R^\times$. Because of this, the statement "f vanishes at $(a_0, \dots, a_n)$" depends only on the point $[a_0 : \dots : a_n]$ in $\mathbb{P}^n(R)$. In other words, a *homogeneous* polynomial equation in $n + 1$ variables has a well-defined solution set in projective n-space.

Let us now return to the equation $y^2 = x^3 - x$ from §1.1 defining the subfunctor $X \subset \mathbb{A}^2$. We can *homogenize* this equation by introducing a new variable w and multiplying each term by the minimal power of w so that the equation becomes homogeneous: this yields the degree 3 homogeneous equation $wy^2 = x^3 - w^2x$. Let $\bar{X} \subset \mathbb{P}^2$ be the subfunctor of solutions to this equation, given by

$$\bar{X}(R) = \{[s : a : b] \in \mathbb{P}^2(R) \mid sb^2 = a^3 - s^2a\}$$

(this makes sense for an arbitrary ring R, although we have only defined it so far when R is a local ring or a principal ideal domain). If we set $s = 1$, we recover precisely the subset $X(R)$ of $\mathbb{A}^2(R) \subset \mathbb{P}^2(R)$:

$$X = \bar{X} \cap \mathbb{A}^2.$$

On the other hand, we can find the solutions "at infinity" by setting $s = 0$, which yields the equation $a^3 = 0$. If k is a field, we see that the only point of $\bar{X}(k)$ with $s = 0$ is $[0 : 0 : 1]$, which is the point at infinity in the vertical direction $(0, 1)$:

$$\bar{X}(k) = X(k) \sqcup \{[0 : 0 : 1]\} \subset \mathbb{P}^2(k).$$

As we surmised in §1.1, $\bar{X}(\mathbb{R})$ is a submanifold of the real projective plane $\mathbb{P}^2(\mathbb{R})$ diffeomorphic to $S^1 \sqcup S^1$, while $\bar{X}(\mathbb{C})$ is a complex submanifold of the complex projective plane $\mathbb{P}^2(\mathbb{C})$, which is diffeomorphic to a torus $S^1 \times S^1$ and biholomorphic to $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}i)$.

One can still define tangent spaces via dual numbers. Let us work out the tangent space at the point at infinity $[0 : 0 : 1] \in \bar{X}(R)$ for a local ring R. Note that $R[\varepsilon]$ is again local and $R[\varepsilon]^\times = R^\times + R\varepsilon$. Given $[s : a : b] \in \bar{X}(R)$, a point $[s + t\varepsilon : a + u\varepsilon : b + v\varepsilon]$ belongs to $\bar{X}(R[\varepsilon])$ if and only if

$$(b^2 + 2sa)t + (-3a^2 + s^2)u + 2sbv = 0.$$

If $[s:a:b] = [0:0:1]$, this reduces to $t = 0$, so that tangent vectors at infinity have the form $[0:u\varepsilon:1+v\varepsilon] = [0 : u\varepsilon : 1]$. Thus, we see that $u \mapsto [0 : u\varepsilon : 1]$ is a bijection from R to the tangent space at $[0 : 0 : 1]$. Since this is a free R-module of rank 1, the point $[0 : 0 : 1]$ is nonsingular.

Similarly, the affine curve $Y \subset \mathbb{A}^2$ defined by the equation $y^2 = x^3 - x + 1$ is compactified to the projective curve $\bar{Y} \subset \mathbb{P}^2$ given by

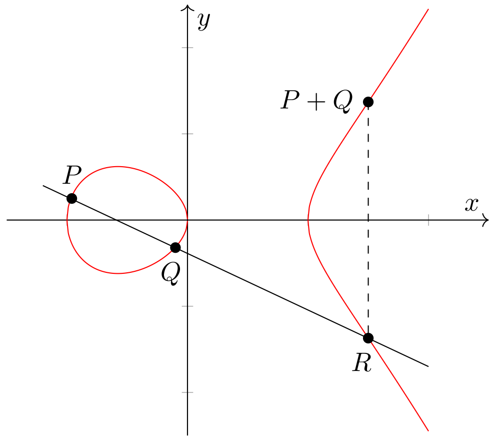
$$\bar{Y}(R) = \{[s : a : b] \in \mathbb{P}^2(R) \mid sb^2 = a^3 - s^2a + s^3\}.$$

For a field k, we have $\bar{Y}(k) = Y(k) \sqcup \{[0:0:1]\}$ as before. The real solution set $\bar{Y}(\mathbb{R})$ is now diffeomorphic to a circle, while $\bar{Y}(\mathbb{C})$ is diffeomorphic to a torus and biholomorphic to $\mathbb{C}/\Lambda$ for some lattice $\Lambda \subset \mathbb{C}$. As in the affine case, $\bar{X}(\mathbb{C})$ and $\bar{Y}(\mathbb{C})$ are diffeomorphic but not biholomorphic. Finally, the nodal cubic N and cuspical cubic C from §1.2 also have compactifications $\bar{N}, \bar{C} \subset \mathbb{P}^2$, which over a field add the single point at infinity $[0 : 0 : 1]$. Over the complex numbers, this has the effect of filling in the puncture.

Away from the primes of bad reduction (which means: when restricted to rings in which these primes are invertible), $\bar{X}$ and $\bar{Y}$ are examples of *elliptic curves*. A remarkable fact is that elliptic curves have an essentially unique group structure. More precisely, if we choose a point e in $\bar{X}(\mathbb{Z}[\frac{1}{2}])$ to serve as the unit element, there is a unique lift

$$\begin{tikzcd} \mathrm{CAlg}_{\mathbb{Z}[\frac{1}{2}]} \arrow[r, "(\bar{X}, e)"] \arrow[dr, dashed, "\exists!"'] & \mathrm{Set}_*. \\ & \mathrm{Grp}. \arrow[u, "\text{forget}"'] \end{tikzcd}$$

Moreover, this lift lands in the subcategory $\mathrm{Ab} \subset \mathrm{Grp}$ of abelian groups. Over the complex numbers, the group structure on $\bar{X}(\mathbb{C})$ is that of the quotient $\mathbb{C}/\Lambda$ (with unit 0). In general, if we take the unit to be the point at infinity, then the group law is determined by the requirement that $P + Q + R = 0$ whenever P, Q, and R lie on a line (if $P = Q$, this means that R lies on the tangent line at P, and if P is the point at infinity, this means that Q and R lie on the same vertical line). Here is an illustration over the real numbers:



The same discussion applies to the elliptic curve $\bar{Y}\colon \mathrm{CAlg}_{\mathbb{Z}[\frac{1}{46}]} \to \mathrm{Set}$ (recall that Y has bad reduction at the primes 2 and 23, and we have to invert both to get an elliptic curve).

1.4. **Fermat curves.** Let $n \in \mathbb{N}$ and consider the homogeneous equation $x^n = y^n + z^n$. Let $X_n(R)$ be its set of solutions in the projective plane $\mathbb{P}^2(R)$ (which we have only defined so far when R is a local ring or principal ideal domain):

$$X_n(R) = \{[a:b:c] \in \mathbb{P}^2(R) \mid a^n = b^n + c^n\}.$$

(i) For $n = 0$, the equation is $1 = 1 + 1$, which holds in a ring R if and only if R is the zero ring. In other words, $X_0(R) = \varnothing$ if $R \neq 0$ and $X_0(0) = \{0\}$. The functor X_0 is the so-called *empty scheme*.

(ii) For $n = 1$, the equation $x = y + z$ cuts out a line in $\mathbb{P}^2$, which is isomorphic to $\mathbb{P}^1$. Indeed, there is a natural bijection

$$\mathbb{P}^1 \xrightarrow{\sim} X_1, \quad [a:b] \mapsto [a+b:a:b].$$

The real projective line $\mathbb{P}^1(\mathbb{R})$ is diffeomorphic to a circle, while the complex projective line $\mathbb{P}^1(\mathbb{C})$, also called the *Riemann sphere*, is diffeomorphic to a sphere.

(iii) For $n = 2$, we have the equation $x^2 = y^2 + z^2$. The solutions to this equation in $\mathbb{Z}^3$ are the so-called *Pythagorean triples*. It turns out that these can be explicitly determined, as we now explain. Call a Pythagorean triple (a,b,c) *primitive* if $a > 0$ and $(a,b,c) = \mathbb{Z}$. Every Pythagorean triple has the form $n(a,b,c)$ for an integer n and a primitive Pythagorean triple (a,b,c) (which are uniquely determined if $n \neq 0$). Sending (a,b,c) to $[a:b:c]$ defines a bijection between the set of primitive Pythagorean triples and $X_2(\mathbb{Z})$. Furthermore, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ induces a bijection $X_2(\mathbb{Z}) \xrightarrow{\sim} X_2(\mathbb{Q})$, with inverse given by clearing denominators (this is true for X_n for all n and reflects the fact that the scheme X_n is *proper*).

Now, for any local $\mathbb{Z}[\frac{1}{2}]$-algebra R (in fact any $\mathbb{Z}[\frac{1}{2}]$-algebra), the map

$$\sigma\colon \mathbb{P}^1(R) \to X_2(R), \quad [a:b] \mapsto [a^2+b^2 : a^2-b^2 : 2ab]$$

is well-defined and bijective. The inverse is given by

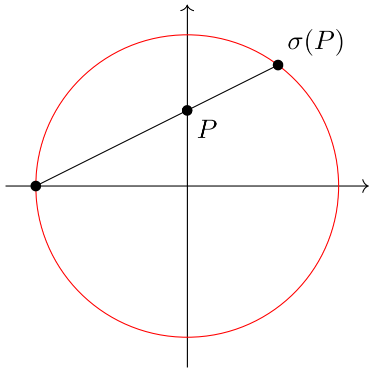
$$[u:v:w] \mapsto \begin{cases} [u+v:w] & \text{if } u+v \in R^\times, \\ [w:u-v] & \text{if } u-v \in R^\times. \end{cases}$$

In particular, for $R = \mathbb{Q}$, σ gives a complete enumeration of all primitive Pythagorean triples, which is known as *Euclid's formula*.

The map $\sigma\colon \mathbb{P}^1(k) \to X_2(k)$ has a concrete geometric interpretation when k is a subfield of $\mathbb{R}$. If $[a:b:c] \in X_2(k)$, then a cannot be zero as then b and c would also be zero (note that this is not true for $k = \mathbb{C}$). Hence,

$$X_2(k) = \{[1:b:c] \in \mathbb{P}^2(k) \mid b^2 + c^2 = 1\} = \{(b,c) \in \mathbb{A}^2(k) \mid b^2 + c^2 = 1\}.$$

In other words, $X_2(k)$ is the set of points on the unit circle in $\mathbb{R}^2$ with coordinates in k. The map σ is then the inverse stereographic projection from the point $(-1,0)$ on the unit circle:



(iv) For $n \geq 3$, Fermat's Last Theorem states that

$$X_n(\mathbb{Q}) = X_n(\mathbb{Z}) = \begin{cases} \{[0:1:-1],[1:0:1],[1:1:0]\} & \text{if } n \text{ is odd,} \\ \{[1:0:\pm 1],[1:\pm 1:0]\} & \text{if } n \text{ is even.} \end{cases}$$

Away from the prime 3, $X_3\colon \mathrm{CAlg}_{\mathbb{Z}[\frac{1}{3}]} \to \mathrm{Set}$ is another example of an elliptic curve, which has a unique group structure with unit element $[0:1:-1]$.

1.5. **Affine, projective, and general schemes.** In §1.1 and §1.2 we saw examples of *affine schemes*, and in §1.3 and §1.4 we saw examples of *projective schemes*. The former are the solutions in affine space of systems of polynomial equations, while the latter are the solutions in projective space of systems of homogeneous polynomial equations. If we allow *inequations* in addition to equations, we obtain the notions of *quasi-affine* and *quasi-projective* schemes. We will study affine schemes in §2 and projective schemes in §3. All of these objects are examples of *schemes*, which we will finally define in §**??**. The following diagram summarizes the situation:

$$\begin{array}{ccccc} \{\text{affine schemes}\} & \subset & \{\text{quasi-affine schemes}\} & & \\ & & \cap & & \\ \{\text{projective schemes}\} & \subset & \{\text{quasi-projective schemes}\} & \subset & \{\text{schemes}\}. \end{array}$$

(Strictly speaking, quasi-projective schemes are defined using polynomials in finitely many variables; for the vertical inclusion to hold, one should either remove this finiteness condition on the quasi-projective side or add it on the quasi-affine side.)

(i) An example of a quasi-affine scheme that is not affine is the *punctured affine n-space* $\mathbb{A}^n - 0$ for $n \geq 2$, which is defined as

$$\mathbb{A}^n - 0\colon \mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \{a \in R^n \mid (a) = R\}.$$

As the notation suggets, this is in a precise sense the complement of 0 in $\mathbb{A}^n$, where 0 is the joint vanishing locus of the coordinate functions $x_1, \dots, x_n$ on $\mathbb{A}^n$.

(ii) An example of a quasi-projective scheme that is neither projective nor quasi-affine is the *punctured projective n-space* $\mathbb{P}^n - 0$ for $n \geq 2$. This is again the complement of 0 in $\mathbb{P}^n$, where 0 is the vanishing locus of the projective coordinates $x_1, \dots, x_n$ on $\mathbb{P}^n$. For a local ring R, we have

$$(\mathbb{P}^n - 0)(R) = \{[a_0 : \dots : a_n] \in \mathbb{P}^n(R) \mid (a_1, \dots, a_n) = R\}.$$

(iii) Schemes that are not quasi-projective are more difficult to come by, but they can be constructed by explicitly gluing affine schemes. The simplest example is the *affine line with doubled origin*, which is the functor

$$\mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \{(f,e) \mid f \in R,\ e \in R/(f), \text{ and } e^2 = e\}.$$

2. Affine geometry

2.1. **Affine spaces.** Affine geometry studies the solutions in affine spaces to systems of polynomial equations, while projective geometry studies the solutions in projective spaces to systems of homogeneous polynomial equations. In both cases, the solutions form a functor $\mathrm{CAlg}_k \to \mathrm{Set}$ from the category of k-algebras to the category of sets. Such functors are the basic objects of algebraic geometry:

Definition 2.1 (Algebraic functor)**.** Let k be a ring. An *algebraic k-functor* is a functor $\mathrm{CAlg}_k \to \mathrm{Set}$. An algebraic $\mathbb{Z}$-functor is simply called an *algebraic functor*. Given an algebraic k-functor X and a k-algebra R, the elements of $X(R)$ are called the *R-valued points* or *R-points* of X.

A basic example is given by affine spaces:

Definition 2.2 (Affine space)**.** Let I be a set. The *affine I-space* over k is the algebraic k-functor

$$\mathbb{A}^I_k \colon \mathrm{CAlg}_k \to \mathrm{Set}, \quad R \mapsto R^I.$$

We simply write $\mathbb{A}^I$ when $k = \mathbb{Z}$. For $n \geq 0$, the *affine n-space* over k is $\mathbb{A}^n_k = \mathbb{A}^{\{1,\dots,n\}}_k$. It is also called the *affine line* if $n = 1$ and the *affine plane* if $n = 2$.

Remark 2.3.
(i) $\mathbb{A}^0_k$ is a final object $*$ of $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.
(ii) $\mathbb{A}^1_k$ is isomorphic to the forgetful functor $\mathrm{CAlg}_k \to \mathrm{Set}$.
(iii) $\mathbb{A}^I_k$ is contravariantly functorial in the set I: a map $f \colon J \to I$ induces a natural transformation $\mathbb{A}^I_k \to \mathbb{A}^J_k$ given by precomposition with f.
(iv) By the universal property of polynomial rings, the functor $\mathbb{A}^I_k$ is represented by the polynomial k-algebra $k[x_i \mid i \in I]$, i.e., there is an isomorphism

$$\mathbb{A}^I_k \simeq \mathrm{Map}(k[x_i \mid i \in I], -) \colon \mathrm{CAlg}_k \to \mathrm{Set}.$$

Indeed, given an I-tuple $(r_i)_{i \in I} \in R^I$, there is a unique k-algebra map $k[x_i \mid i \in I] \to R$ sending x_i to r_i.

2.2. **Presheaves.** We start with some categorical preliminaries on set-valued functors, also known as *presheaves*.

Definition 2.4 (Presheaves)**.** Let $\mathcal{C}$ be a category. A *presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. We denote by

$$\mathrm{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Set})$$

the category of presheaves on $\mathcal{C}$. More generally, given an arbitrary category $\mathcal{E}$, an *$\mathcal{E}$-valued presheaf* on $\mathcal{C}$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{E}$.

For example, an algebraic k-functor is exactly a presheaf on $\mathrm{CAlg}^{\mathrm{op}}_k$.

Remark 2.5. The category $\mathrm{P}(\mathcal{C})$ always admits limits and colimits, which are computed "pointwise" in the category of sets. Many properties of the category of sets are thereby inherited by the category of presheaves $\mathrm{P}(\mathcal{C})$, such as the fact that filtered colimits commute with finite limits, the fact the monomorphisms and epimorphisms are effective, etc.

Definition 2.6 (Yoneda embedding)**.** Let $\mathcal{C}$ be a category. The *Yoneda embedding* of $\mathcal{C}$ is the functor

$$よ \colon \mathcal{C} \to \mathrm{P}(\mathcal{C}), \quad よ(X) = \mathrm{Map}(-, X) \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}.$$

A presheaf on $\mathcal{C}$ is called *representable* if it lies in the essential image of $よ$.

Notation 2.7 (The functor Spec)**.** When $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}_k$, the Yoneda embedding is denoted by

$$\mathrm{Spec} \colon \mathrm{CAlg}^{\mathrm{op}}_k \to \mathrm{P}(\mathrm{CAlg}^{\mathrm{op}}) = \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}).$$

Thus, $\mathrm{Spec}(A)$ is the algebraic k-functor represented by the k-algebra A:

$$\mathrm{Spec}(A)(R) = \mathrm{Map}(A, R).$$

For example, $\mathbb{A}^I_k \simeq \mathrm{Spec}(k[x_i \mid i \in I])$.

Definition 2.8 (Category of elements)**.** Let $\mathcal{C}$ be a category and let $F \in \mathrm{P}(\mathcal{C})$ be a presheaf on $\mathcal{C}$. The *category of elements* $\mathrm{El}(F)$ of F is defined by the cartesian square

$$\begin{tikzcd} \mathrm{El}(F) \arrow[r] \arrow[d] & (\mathrm{Set}_*)^{\mathrm{op}} \arrow[d, "\text{forget}"] \\ \mathcal{C} \arrow[r, "F"] & \mathrm{Set}^{\mathrm{op}}. \end{tikzcd}$$

It is also denoted by $\int F$. Explicitly, objects of $\mathrm{El}(F)$ are pairs (X, x) with $X \in \mathcal{C}$ and $x \in F(X)$, and morphisms $(X, x) \to (Y, y)$ are morphisms $f \colon X \to Y$ in $\mathcal{C}$ such that $f^*(y) = x$.

Theorem 2.9 (Properties of the Yoneda embedding)**.** *Let $\mathcal{C}$ be a category.*
(i) (The Yoneda Lemma) *Let $X \in \mathcal{C}$ and $F \in \mathrm{P}(\mathcal{C})$. Then the map*

$$\mathrm{Map}(よ(X), F) \to F(X), \quad f \mapsto f(\mathrm{id}_X),$$

is a bijection with inverse $x \mapsto ((f \colon Y \to X) \mapsto f^*(x))$.

(ii) *The Yoneda embedding* $\text{よ}\colon \mathcal{C} \to \mathrm{P}(\mathcal{C})$ *is fully faithful.*
(iii) *The Yoneda embedding* $\text{よ}\colon \mathcal{C} \to \mathrm{P}(\mathcal{C})$ *preserves all limits that exist in* $\mathcal{C}$*.*
(iv) *Every presheaf* $F \in \mathrm{P}(\mathcal{C})$ *is canonically a colimit of representable presheaves:*

$$\operatorname{colim}\left(\mathrm{El}(F) \xrightarrow{\text{forget}} \mathcal{C} \xrightarrow{\text{よ}} \mathrm{P}(\mathcal{C})\right) \xrightarrow{\sim} F.$$

For the next two statements, assume that $\mathcal{C}$ *is small.*

(v) (Universal property of よ) *Let* $\mathcal{E}$ *be a cocomplete category. Then the functor*

$$\text{よ}^*\colon \mathrm{Fun}^{\mathrm{colim}}(\mathrm{P}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{E}),$$

is an equivalence of categories, where $\mathrm{Fun}^{\mathrm{colim}}$ *denotes the category of colimit-preserving functor; the inverse is given by left Kan extension along* よ*. In particular, any functor* $\mathcal{C} \to \mathcal{E}$ *extends uniquely (up to unique isomorphism) to a colimit-preserving functor* $\mathrm{P}(\mathcal{C}) \to \mathcal{E}$*.*
(vi) *If* $\mathcal{E}$ *is any category, then any colimit-preserving functor* $K\colon \mathrm{P}(\mathcal{C}) \to \mathcal{E}$ *has a right adjoint* $\mathcal{E} \to \mathrm{P}(\mathcal{C})$ *given by* $e \mapsto \mathrm{Map}(K(\text{よ}(-)), e)$*.*

Remark 2.10.

(i) By the Yoneda Lemma, the category of elements of a presheaf $F \in \mathrm{P}(\mathcal{C})$ can equivalently be described as the pullback

$$\begin{tikzcd} \mathrm{El}(F) \arrow[r] \arrow[d] & \mathrm{P}(\mathcal{C})_{/F} \arrow[d, "\text{forget}"] \\ \mathcal{C} \arrow[r, "\text{よ}"] & \mathrm{P}(\mathcal{C}), \end{tikzcd}$$

whose objects are pairs (X, x) with $X \in \mathcal{C}$ and $x\colon \text{よ}(X) \to F$. Theorem 2.9(iv) then says that every presheaf is the colimit of all representable presheaves mapping to it.
(ii) By the full faithfulness of the Yoneda embedding, we have $\mathrm{El}(\text{よ}(X)) \simeq \mathcal{C}_{/X}$. Together with Theorem 2.9(iv), this shows that a presheaf is representable if and only if its category of elements has a final object (which is then the representing object).

Corollary 2.11. *Let* $F, G \in \mathrm{P}(\mathcal{C})$ *be presheaves. Then*

$$\mathrm{Map}(F, G) = \lim_{(X,x)\in \mathrm{El}(F)} G(X).$$

Example 2.12. By Theorem 2.9(iii), the functor

$$\mathrm{Spec}\colon \mathrm{CAlg}_k^{\mathrm{op}} \to \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$$

preserves all limits. Moreover, limits in $\mathrm{CAlg}_k^{\mathrm{op}}$ are colimits in CAlg_k. For example, k is the initial object of CAlg_k, so that $\mathrm{Spec}(k)$ is the final object of $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$. The coproduct of two k-algebras A and B is the tensor product $A \otimes_k B$, so that

$$\mathrm{Spec}(A \otimes_k B) \simeq \mathrm{Spec}(A) \times \mathrm{Spec}(B)$$

in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$. More generally, the pushout of a diagram $A \leftarrow C \to B$ in CAlg_k is the relative tensor product $A \otimes_C B$, so that

$$\mathrm{Spec}(A \otimes_C B) \simeq \mathrm{Spec}(A) \times_{\mathrm{Spec}(C)} \mathrm{Spec}(B).$$

Remark 2.13 (Algebraic structures on presheaves)**.** Algebraic objects like monoids, groups, abelian groups, rings, modules over a ring, etc., make sense in any category with finite products. In categories of presheaves, since finite products are computed objectwise, algebraic objects are the same as presheaves valued in the category of algebraic objects of the same type in Set. For example, abelian group objects in presheaves are the same as presheaves of abelian groups:

$$\mathrm{Ab}(\mathrm{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}).$$

Thus, given a presheaf $F \in \mathrm{P}(\mathcal{C})$, equipping F with an abelian group structure is equivalent to lifting F along the forgetful functor $\mathrm{Ab} \to \mathrm{Set}$:

$$\begin{tikzcd} & \mathrm{Ab} \arrow[d, "\text{forget}"] \\ \mathcal{C}^{\mathrm{op}} \arrow[r, "F"'] \arrow[ur, dashed] & \mathrm{Set}. \end{tikzcd}$$

2.3. **Polynomial equations.**

Definition 2.14 (System of polynomial equations)**.** Let k be a ring and let I and J be sets. A *system of J polynomial equations in I variables* over k is a J-tuple $(f_j)_{j\in J}$ in the polynomial ring $k[x_i \mid i \in I]$. We denote by (Σ) the ideal in $k[x_i \mid i \in I]$ generated by $(f_j)_{j\in J}$ and by $k[\Sigma]$ the k-algebra $k[x_i \mid i \in I]/(\Sigma)$.

Remark 2.15. Every k-algebra R is isomorphic to $k[\Sigma]$ for some system of polynomial equations Σ. A choice of isomorphism $R \simeq k[\Sigma]$ is exactly a *presentation* of R by generators and relations.

Given a system of polynomial equations over k, we can consider its solutions in any k-algebra. To that end, recall that there is, for any k-algebra R, an *evaluation map*

$$k[x_i \mid i \in I] \times R^I \to R, \quad (f, a) \mapsto f(a),$$

which is defined as follows: for each $a \in R^I$, $f \mapsto f(a)$ is the unique k-algebra map $k[x_i \mid i \in I] \to R$ sending x_i to a_i.

Definition 2.16 (Vanishing locus)**.** Let $F \subset k[x_i \mid i \in I]$ be a subset. The *vanishing locus* of F in $\mathbb{A}^I_k$ is the subfunctor $\mathrm{V}(F) \subset \mathbb{A}^I_k$ given by

$$\mathrm{V}(F)(R) = \{a \in R^I \mid f(a) = 0 \text{ for all } f \in F\} \subset R^I.$$

This is indeed a subfunctor: for any k-algebra map $R \to S$, the induced map $R^I \to S^I$ sends $\mathrm{V}(F)(R)$ to $\mathrm{V}(F)(S)$.

Remark 2.17. It is clear that the vanishing locus of F depends only on the ideal generated by F: if $(F) = (F')$, then $\mathrm{V}(F) = \mathrm{V}(F')$. We will see below that the converse also holds (Corollary 2.25).

Definition 2.18 (Solution functor)**.** Let $\Sigma = (f_j)_{j\in J}$ be a system of J polynomial equations in I variables over k. Its *solution functor* $\mathrm{Sol}_\Sigma \colon \mathrm{CAlg}_k \to \mathrm{Set}$ is the vanishing locus of $\{f_j \mid j \in J\}$ in $\mathbb{A}^I_k$:

$$\mathrm{Sol}_\Sigma = \mathrm{V}(\{f_j \mid j \in J\}) \subset \mathbb{A}^I_k.$$

By the universal property of polynomial rings, there is a one-to-one correspondence between systems of J polynomial equations in I variables and k-algebra maps

$$k[x_j \mid j \in J] \to k[x_i \mid i \in I].$$

By the Yoneda lemma, these are in turn equivalent to natural transformations

$$\mathbb{A}^I_k \to \mathbb{A}^J_k \colon \mathrm{CAlg}_k \to \mathrm{Set}.$$

Unraveling these equivalences, the map $\mathbb{A}^I_k \to \mathbb{A}^J_k$ corresponding to a system $\Sigma = (f_j)_{j\in J}$ is given on a k-algebra R by

$$R^I \to R^J, \quad a \mapsto (f_j(a))_{j\in J}.$$

By definition, the solution functor Sol_Σ is the kernel of this map, i.e., there is a pullback square

$$\begin{tikzcd} \mathrm{Sol}_\Sigma \arrow[r, hook] \arrow[d] & \mathbb{A}^I_k \arrow[d] \\ 0 \arrow[r, hook] & \mathbb{A}^J_k, \end{tikzcd} \tag{2.19}$$

where 0 is the subfunctor of $\mathbb{A}^J_k$ given by $0(R) = \{0\} \subset R^J$.

Definition 2.20 (Affine scheme)**.** A functor $\mathrm{CAlg}_k \to \mathrm{Set}$ is called an *affine k-scheme* if it is isomorphic to Sol_Σ for some system of polynomial equations Σ over k. We denote by $\mathrm{Aff}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$ the full subcategory spanned by the affine k-schemes. An *affine scheme* is an affine $\mathbb{Z}$-scheme.

Example 2.21. The affine I-space $\mathbb{A}^I_k$ is an affine k-scheme, as it is the solution functor of the empty system of equations in I variables.

Lemma 2.22. *Let Σ be a system of polynomial equations over k. Then the solution functor* Sol_Σ *is represented by the k-algebra $k[\Sigma]$, i.e., there is an isomorphism*

$$\mathrm{Sol}_\Sigma \simeq \mathrm{Spec}(k[\Sigma]) \colon \mathrm{CAlg}_k \to \mathrm{Set}.$$

Theorem 2.23 (Characterization of affine schemes)**.** *Let k be a ring. The following conditions are equivalent for an algebraic k-functor $X \colon \mathrm{CAlg}_k \to \mathrm{Set}$:*

(i) *X is an affine k-scheme.*
(ii) *X is representable, i.e., isomorphic to* $\mathrm{Spec}(A)$ *for some k-algebra A.*

(iii) *X preserves limits and is accessible*[1].

Corollary 2.24. *The Yoneda embedding of* $\mathrm{CAlg}_k^{\mathrm{op}}$ *induces an equivalence of categories*

$$\mathrm{Spec}\colon \mathrm{CAlg}_k^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Aff}_k \subset \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}).$$

Under this equivalence, the affine k-scheme Sol_Σ *corresponds to the k-algebra* $k[\Sigma]$.

Under the equivalence of Corollary 2.24, the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ of affine k-schemes corresponds to the quotient map $k[x_i \mid i \in I] \twoheadrightarrow k[\Sigma]$. This implies the following result:

Corollary 2.25 (Functorial Nullstellensatz)**.** *Sending a subset $F \subset k[x_i \mid i \in I]$ to its vanishing locus* $\mathrm{V}(F) \subset \mathbb{A}^I_k$ *induces an order-reversing bijection*

$$\mathrm{V}\colon \{\text{ideals in } k[x_i \mid i \in I]\} \xrightarrow{\sim} \{\text{vanishing loci in } \mathbb{A}^I_k\}.$$

Example 2.26. Consider the following systems of polynomial equations over $\mathbb{R}$ in one variable:

$$\Sigma_1 = (x^2+1), \quad \Sigma_2 = ((x^2+1)^2), \quad \Sigma_3 = (x^2+x+1), \quad \Sigma_4 = (x^4+1).$$

Then:

$$\begin{array}{llll} \mathrm{Sol}_{\Sigma_1}(\mathbb{R}) = \varnothing & \mathrm{Sol}_{\Sigma_2}(\mathbb{R}) = \varnothing & \mathrm{Sol}_{\Sigma_3}(\mathbb{R}) = \varnothing & \mathrm{Sol}_{\Sigma_4}(\mathbb{R}) = \varnothing, \\ \mathrm{Sol}_{\Sigma_1}(\mathbb{C}) = \{\pm i\} & \mathrm{Sol}_{\Sigma_2}(\mathbb{C}) = \{\pm i\} & \mathrm{Sol}_{\Sigma_3}(\mathbb{C}) = \{\zeta_3, \bar\zeta_3\} & \mathrm{Sol}_{\Sigma_4}(\mathbb{C}) = \{\pm\zeta_8, \pm\bar\zeta_8\}, \end{array}$$

where $\zeta_n = \exp\big(\frac{2\pi i}{n}\big) \in \mathbb{C}$. All four equations have the same solutions in $\mathbb{R}$. However, as the four ideals (Σ_i) in $\mathbb{R}[x]$ are pairwise distinct, they define four different subfunctors of $\mathbb{A}^1_{\mathbb{R}}$ by Corollary 2.25. The solutions in $\mathbb{C}$ distinguish them, except for Sol_{Σ_1} and Sol_{Σ_2}. To see that $\mathrm{Sol}_{\Sigma_1} \neq \mathrm{Sol}_{\Sigma_2}$ as subfunctors of $\mathbb{A}^1_{\mathbb{R}}$, we can compute the solutions in the $\mathbb{R}$-algebra $\mathbb{C}[\varepsilon]$ of dual complex numbers (where $\varepsilon^2 = 0$):

$$\mathrm{Sol}_{\Sigma_1}(\mathbb{C}[\varepsilon]) = \{\pm i\}, \quad \mathrm{Sol}_{\Sigma_2}(\mathbb{C}[\varepsilon]) = \{\pm i + a\varepsilon \mid a \in \mathbb{C}\}.$$

On the other hand, the associated $\mathbb{R}$-algebras are

$$\mathbb{R}[\Sigma_1] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_2] \simeq \mathbb{C}[\varepsilon], \quad \mathbb{R}[\Sigma_3] \simeq \mathbb{C}, \quad \mathbb{R}[\Sigma_4] \simeq \mathbb{C}\times\mathbb{C}.$$

By Lemma 2.22, Sol_{Σ_1} and Sol_{Σ_3} are both isomorphic to $\mathrm{Spec}(\mathbb{C})$. The different ideals (Σ_1) and (Σ_3) correspond to two different embeddings of the affine $\mathbb{R}$-scheme $\mathrm{Spec}(\mathbb{C})$ into $\mathbb{A}^1_{\mathbb{R}}$, and the systems Σ_1 and Σ_3 themselves are two different presentations of the $\mathbb{R}$-algebra $\mathbb{C}$.

Remark 2.27. In summary, given a system of polynomial equations Σ over k, we have the following relations between Σ and Sol_Σ:

(i) The data of the pullback square (2.19) is equivalent to the data of Σ itself.
(ii) The data of the embedding $\mathrm{Sol}_\Sigma \hookrightarrow \mathbb{A}^I_k$ is equivalent to the data of the ideal (Σ) in the polynomial ring $k[x_i \mid i \in I]$.
(iii) The data of the affine k-scheme Sol_Σ alone is equivalent to the data of the k-algebra $k[\Sigma]$.

This can be compared with the following types of data in differential geometry:

(i) A smooth manifold M given as the vanishing locus of a smooth function $\mathbb{R}^n \to \mathbb{R}^m$.
(ii) A smooth manifold M given as a closed submanifold of $\mathbb{R}^n$.
(iii) A smooth manifold M.

Smooth manifolds are the basic objects of interest in differential geometry. Embedding a manifold M into a Euclidean space or realizing it as the vanishing locus of a function are often useful ways to understand M, but we do not consider this additional data to be part of the manifold M itself. The situation in algebraic geometry is entirely similar: the basic objects of interest are affine schemes. Any affine scheme X can be embedded into an affine space ($X \hookrightarrow \mathbb{A}^I$) or realized as the solution functor of a system of polynomial equations ($X \simeq \mathrm{Sol}_\Sigma$), but this data is not part of the affine scheme X itself.

A key difference between differential geometry and algebraic geometry is that it is much easier to embed smooth manifolds into $\mathbb{R}^n$ than it is to embed schemes into $\mathbb{A}^n$. In fact, the former is always possible under mild technical assumptions[2] (which are usually taken as part of the definition of smooth manifold), but many interesting schemes are not affine. For example, the real projective space $\mathbb{P}^n(\mathbb{R})$ can be embedded the Euclidean space $\mathbb{R}^{2n}$, but we will see that the algebraic projective space $\mathbb{P}^n$ with $n \geq 1$ cannot be embedded in $\mathbb{A}^N$ for any N.

[1] Accessibility of X is technical condition saying that X is a *small* colimit of representables. It is equivalent to the condition that X preserves κ-filtered colimits for some infinite cardinal κ, which is usually easy to check in practice.

[2] namely: Hausdorff, second countable, and of bounded dimension

Remark 2.28 (Systems of linear equations)**.** Let us spell out the analogy with linear algebra. A system Λ of J linear equations in I variables over a ring k is a J-indexed family in the free k-module $k^{(I)}$, or equivalently a k-linear map $k^{(J)} \to k^{(I)}$. If $(a_{ij})_{i\in I, j\in J}$ is the corresponding $I \times J$-matrix, a solution to Λ in a k-module M is a family $(m_i)_{i\in I}$ in M such that $\sum_{i\in I} a_{ij}m_i = 0$ for all $j \in J$. This defines a *solution functor*

$$\mathrm{Sol}_\Lambda \colon \mathrm{Mod}_k \to \mathrm{Set}.$$

Unraveling the definitions, $\mathrm{Sol}_\Lambda(M)$ is exactly the kernel of the map $M^I \to M^J$, obtained by applying $\mathrm{Map}(-, M)$ to the given map $k^{(J)} \to k^{(I)}$. It follows that $\mathrm{Sol}_\Lambda \simeq \mathrm{Map}(C, -)$, where C is the cokernel of $k^{(J)} \to k^{(I)}$. Thus, we can think of a system of linear equations over k as a k-module C equipped with a presentation, and its solution functor as the k-module C itself.

2.4. **Examples of affine schemes.**

Example 2.29 (The final scheme)**.** The constant functor $\mathrm{CAlg} \to \mathrm{Set}$ sending every ring to a one-point set is isomorphic to $\mathrm{Spec}(\mathbb{Z})$ and hence is an affine scheme. This is the final object of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$.

Example 2.30 (The empty scheme)**.** The functor

$$\mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto \begin{cases} \varnothing & \text{if } R \neq 0, \\ * & \text{if } R = 0, \end{cases}$$

is an affine scheme, isomorphic to $\mathrm{Spec}(0)$. It is called the *empty scheme* and denoted by $\varnothing$. Note that $\varnothing$ is the initial object of Aff, but it *not* the initial object of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, which is the constant functor with value $\varnothing$.

Example 2.31 (The idempotent classifier)**.** Let $\mathrm{Idem} \colon \mathrm{CAlg} \to \mathrm{Set}$ be the functor sending R to the set of idempotent elements of R. Then Idem is an affine scheme, isomorphic to $\mathrm{Spec}(\mathbb{Z} \times \mathbb{Z})$. Indeed, there is a bijection

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{Z} \times \mathbb{Z}, R) &\xrightarrow{\sim} \mathrm{Idem}(R), \\ \varphi &\mapsto \varphi(1, 0), \end{aligned}$$

which is natural in $R \in \mathrm{CAlg}$.

Example 2.32 (The multiplicative group)**.** The functor $\mathbb{G}_\mathrm{m} \colon \mathrm{CAlg} \to \mathrm{Ab}$ sending R to the group of units $R^\times$ is called the *multiplicative group*. It is an *affine group scheme*, meaning that the composition

$$\mathrm{CAlg} \xrightarrow{\mathbb{G}_\mathrm{m}} \mathrm{Ab} \xrightarrow{\text{forget}} \mathrm{Set}$$

is an affine scheme. Indeed, it is isomorphic to $\mathrm{Spec}(\mathbb{Z}[u^{\pm 1}])$: for every ring R, there is a bijection

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}}(\mathbb{Z}[u^{\pm 1}], R) &\xrightarrow{\sim} R^\times, \\ \varphi &\mapsto \varphi(u). \end{aligned}$$

Example 2.33 (The additive group)**.** The functor $\mathbb{G}_\mathrm{a} \colon \mathrm{CAlg} \to \mathrm{Ab}$ sending R to the underlying group $(R, +)$ is called the *additive group*. The composition

$$\mathrm{CAlg} \xrightarrow{\mathbb{G}_\mathrm{a}} \mathrm{Ab} \xrightarrow{\text{forget}} \mathrm{Set}$$

is simply the forgetful functor, also known as the affine scheme $\mathbb{A}^1$. Hence $\mathbb{G}_\mathrm{a}$ is an affine group scheme.

Example 2.34 (The matrix ring)**.** Let $n \geq 0$ and let $\mathrm{Mat}_n \colon \mathrm{CAlg} \to \mathrm{Alg}$ be the functor sending R to the associative ring of $n \times n$ matrices over R. This is an associative ring object in affine schemes. Indeed, since a matrix over R is simply a family of n^2 elements of R, the composition

$$\mathrm{CAlg} \xrightarrow{\mathrm{Mat}_n} \mathrm{Alg} \xrightarrow{\text{forget}} \mathrm{Set}$$

is isomorphic to $\mathbb{A}^{n^2}$ and hence is an affine scheme.

Example 2.35 (The general linear group)**.** Let $n \geq 0$ and let $\mathrm{GL}_n \colon \mathrm{CAlg} \to \mathrm{Grp}$ be the functor sending R to the group $\mathrm{GL}_n(R)$ of invertible $n \times n$ matrices. Then GL_n is an affine group scheme. Indeed, let $A = \mathbb{Z}[x_{ij} \mid (i,j) \in \{1, \dots, n\}^2]$ be the ring representing Mat_n, which contains the universal $n \times n$ matrix $X = (x_{ij})_{i,j}$. A matrix $M \in \mathrm{Mat}_n(R)$ is invertible if and only if its determinant $\det(M) \in R$ is a unit. Hence, for any ring R, there is an isomorphism

$$\begin{aligned} \mathrm{Map}_{\mathrm{CAlg}}(A_{\det(X)}, R) &\xrightarrow{\sim} \mathrm{GL}_n(R), \\ \varphi &\mapsto (\varphi(x_{ij}))_{i,j}, \end{aligned}$$

so that $\mathrm{GL}_n \simeq \mathrm{Spec}(A_{\det(X)})$.

Example 2.36 (The special linear group)**.** Let $n \geq 0$ and let $\mathrm{SL}_n\colon \mathrm{CAlg} \to \mathrm{Grp}$ be the functor sending R to the special linear group $\mathrm{SL}_n(R)$ of $n \times n$ matrices with determinant 1. Let $X \in \mathrm{Mat}_n(A)$ be the universal $n \times n$ matrix as in Example 2.35. We then have

$$\mathrm{Map}_{\mathrm{CAlg}}(A/(\det(X)-1), R) \xrightarrow{\sim} \mathrm{SL}_n(R),$$
$$\varphi \mapsto (\varphi(x_{ij}))_{i,j},$$

so that $\mathrm{SL}_n \simeq \mathrm{Spec}(A/(\det(X)-1))$. The subfunctor inclusions $\mathrm{SL}_n \subset \mathrm{GL}_n \subset \mathrm{Mat}_n$ correspond to the ring maps

$$A \hookrightarrow A_{\det(X)} \twoheadrightarrow A/(\det(X)-1).$$

Example 2.37 (Affine scheme associated with a module)**.** Let k be a ring and let M be a k-module. Consider the functor $\mathbb{A}(M)\colon \mathrm{CAlg}_k \to \mathrm{Mod}_k$ defined by

$$\mathbb{A}(M)(R) = \{k\text{-linear maps } M \to R\} = (M \otimes_k R)^\vee.$$

Then $\mathbb{A}(M)$ is a k-module object in affine k-schemes. Indeed, if $\mathrm{Sym}_k(M)$ is the free k-algebra on M, there is a bijection

$$\mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}_k(M), R) \xrightarrow{\sim} \mathbb{A}(M)(R),$$
$$\varphi \mapsto \varphi|M,$$

so that $\mathbb{A}(M) \simeq \mathrm{Spec}(\mathrm{Sym}_k(M))$. Affine spaces are a special case of this construction: $\mathbb{A}^I_k \simeq \mathbb{A}(k^{(I)})$.

Remark 2.38. Let k be a ring and let M be a k-module. Given Example 2.37, it is tempting to consider the following "predual" of $\mathbb{A}(M)$: define $\mathbb{A}^\vee(M)\colon \mathrm{CAlg}_k \to \mathrm{Mod}_k$ by

$$\mathbb{A}^\vee(M)(R) = M \otimes_k R.$$

There is a canonical map $\mathbb{A}^\vee(M^\vee) \to \mathbb{A}(M)$, which is an isomorphism if and only if M is a vector space (Definition **??**). Otherwise, $\mathbb{A}^\vee(M)$ does not preserve limits and hence is not an affine k-scheme (in fact, it is not even a scheme). For that reason, the functor $\mathbb{A}^\vee(M)$ is rarely used.

2.5. **Base change.** Given a ring map $\varphi\colon k \to k'$, we can transform any system of polynomial equations Σ over k into a system $\varphi^*(\Sigma)$ over k' by applying φ to all the coefficients. More generally, many types of data over k can be transformed into data over k' using φ, a process known as *base change*, *change of coefficients*, or *extension of scalars*. Other examples are the functor $\varphi^*\colon \mathrm{Mod}_k \to \mathrm{Mod}_{k'}$ sending a k-module M to the k'-module $M \otimes_k k'$, and the functor $\varphi^*\colon \mathrm{CAlg}_k \to \mathrm{CAlg}_k'$ sending a k-algebra A to the k'-algebra $A \otimes_k k'$. Unraveling these constructions, we see that there is a canonical isomorphism of k'-algebras

$$k'[\varphi^*(\Sigma)] \simeq \varphi^*(k[\Sigma]).$$

In this section, we investigate the related process of transforming an algebraic k-functor into an algebraic k'-functor.

Theorem 2.39 (Functoriality of presheaves)**.** *Let $u\colon \mathcal{C} \to \mathcal{D}$ be a functor, and let*

$$u^*\colon \mathrm{P}(\mathcal{D}) \to \mathrm{P}(\mathcal{C}), \quad F \mapsto F \circ u,$$

be the "restriction along u" functor.

(i) *u^* admits a left adjoint $u_\sharp$ and a right adjoint u_* given by*

$$u_\sharp(F)(d) = \mathrm{colim}\left((\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{d/})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}\right) = \operatorname*{colim}_{d \to u(c)} F(c),$$
$$u_*(F)(d) = \lim\left((\mathcal{C} \times_{\mathcal{D}} \mathcal{D}_{/d})^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}} \xrightarrow{F} \mathrm{Set}\right) = \lim_{u(c) \to d} F(c),$$

provided these colimits and limits exist (e.g., if $\mathcal{C}$ is small).

(ii) *The functor $u_\sharp$ extends u: there is a canonical isomorphism*

$$u_\sharp \circ よ_{\mathcal{C}} \simeq よ_{\mathcal{D}} \circ u.$$

(iii) *If u is fully faithful, then $u_\sharp$ and u_* are fully faithful.*

(iv) *If u is a localization, then u^* is fully faithful.*

(v) *If the functor u has a left adjoint u_L (resp. a right adjoint u_R), then there is a canonical isomorphism $u_\sharp \simeq u_L^*$ (resp. $u_* \simeq u_R^*$).*

Remark 2.40.

(i) Given $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, the presheaves $u_\sharp(F)$ and $u_*(F)$ are special cases of Kan extensions: $u_\sharp(F)$ is the left Kan extension of F along u^{op}, and $u_*(F)$ is the right Kan extension of F along u^{op}.

(ii) By the universal property of $よ_{\mathcal{C}}$, the functor $u_\sharp$ is the *unique* colimit-preserving extension of u (up to unique isomorphism). However, there is no analogous characterization of u_*.

Corollary 2.41. *Let $\mathcal{C}$ be a category, let $Y \to X$ be a morphism in $\mathcal{C}$, and let $u\colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ be the forgetful functor.*

(i) *The functor $u^*\colon \mathrm{P}(\mathcal{C}_{/X}) \to \mathrm{P}(\mathcal{C}_{/Y})$ has a left adjoint $u_\sharp$ given by*

$$u_\sharp(F)(U \to X) = \coprod_{\substack{\text{maps } U \to Y \\ \text{over } X}} F(U \to Y).$$

(ii) *If pullbacks along $Y \to X$ exist in $\mathcal{C}$, the functor $u^*\colon \mathrm{P}(\mathcal{C}_{/X}) \to \mathrm{P}(\mathcal{C}_{/Y})$ has a right adjoint u_* given by*

$$u_*(F)(U \to X) = F(U \times_X Y \to Y).$$

If k is a ring, then $\mathrm{CAlg}_k \simeq \mathrm{CAlg}_{k/}$ and hence $\mathrm{CAlg}_k^{\mathrm{op}} \simeq (\mathrm{CAlg}^{\mathrm{op}})_{/k}$. Using this identification, we obtain the following special case of Corollary 2.41 with $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$:

Corollary 2.42. *Let $\varphi\colon k \to k'$ be a ring map. Then there is a triple of adjoint functors*

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{CAlg}_{k'}, \mathrm{Set}) \arrow[r, bend left, "\varphi_\sharp"] \arrow[r, bend right, "\varphi_*"'] & \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}), \arrow[l, "\varphi^*"'] \end{tikzcd}$$

where:

- *φ^* is precomposition with the forgetful functor $\mathrm{CAlg}_{k'} \to \mathrm{CAlg}_k$, and it is the unique colimit-preserving extension of $\varphi^*\colon \mathrm{CAlg}_k^{\mathrm{op}} \to \mathrm{CAlg}_{k'}^{\mathrm{op}}$;*
- *φ_* is precomposition with $\varphi^*\colon \mathrm{CAlg}_k \to \mathrm{CAlg}_{k'}$;*
- *$\varphi_\sharp$ is given by*

$$\varphi_\sharp(X)(A) = \coprod_{\substack{k'\text{-algebra} \\ \text{structures on } A}} X(A),$$

and it is the unique colimit-preserving extension of the forgetful functor $\mathrm{CAlg}_{k'}^{\mathrm{op}} \to \mathrm{CAlg}_k^{\mathrm{op}}$.

Definition 2.43. Let $\varphi\colon k \to k'$ be a ring map, giving rise to the adjoint triple of Corollary 2.42.

(i) The functor φ^* is called *base change* or *extension of scalars* along φ and is also denoted by $X \mapsto X_{k'}$.

(ii) The functor φ_* is called *Weil restriction* or *restriction of scalars* along φ and is also denoted by R_φ or $\mathrm{R}_{k'/k}$.

Remark 2.44. Corollary 2.42 says in particular that the functors φ^* and $\varphi_\sharp$ preserve affine schemes:

(i) For a k-algebra A, $\varphi^*(\mathrm{Spec}(A)) \simeq \mathrm{Spec}(A \otimes_k k')$ in $\mathrm{Fun}(\mathrm{CAlg}_{k'}, \mathrm{Set})$. Hence, for any system of polynomial equations Σ over k, $\varphi^*(\mathrm{Sol}_\Sigma) \simeq \mathrm{Sol}_{\varphi^*(\Sigma)}$.

(ii) For a k'-algebra B, $\varphi_\sharp(\mathrm{Spec}(B)) \simeq \mathrm{Spec}(B)$ in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.

On the other hand, the functor φ_* does not always preserve affine schemes.

Example 2.45. Consider the affine scheme $\mathbb{G}_{\mathrm{m}}\colon \mathrm{CAlg} \to \mathrm{Set}$, $R \mapsto R^\times$. Let $\mathbb{G}_{\mathrm{m},\mathbb{C}}\colon \mathrm{CAlg}_{\mathbb{C}} \to \mathrm{Set}$ be its base change to $\mathbb{C}$, i.e., its restriction along the forgetful functor $\mathrm{CAlg}_{\mathbb{C}} \to \mathrm{CAlg}$. The Weil restriction $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})\colon \mathrm{CAlg}_{\mathbb{R}} \to \mathrm{Set}$ is given by

$$\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})(A) = \mathbb{G}_{\mathrm{m},\mathbb{C}}(A \otimes_{\mathbb{R}} \mathbb{C}) = (A \otimes_{\mathbb{R}} \mathbb{C})^\times.$$

One can check that $\mathrm{R}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m},\mathbb{C}})$ is an affine $\mathbb{R}$-scheme: for any $\mathbb{R}$-algebra A, there is a bijection

$$\begin{aligned} \mathrm{Map}\big(\mathbb{R}[x,y,z,w]/(xz - yw - 1, yz + xw), A\big) &\xrightarrow{\sim} (A \otimes_{\mathbb{R}} \mathbb{C})^\times, \\ \varphi &\mapsto \varphi(x) + i\varphi(y). \end{aligned}$$

These equations ensure that $\varphi(x) + i\varphi(y)$ is inverse to $\varphi(z) + i\varphi(w)$. More generally, if $k \subset k'$ is a finite field extension, one can show that Weil restriction $\mathrm{R}_{k'/k}$ sends affine k'-schemes to affine k-schemes.

A fundamental property of sets is that *maps of sets* are equivalent to *families of sets*: there is an equivalence of categories

$$\mathrm{Ar}(\mathrm{Set}) \simeq \mathrm{Fam}(\mathrm{Set}),$$

where $\mathrm{Ar}(\mathrm{Set}) = \mathrm{Fun}(\{0 \to 1\}, \mathrm{Set})$ is the arrow category of Set and $\mathrm{Fam}(\mathrm{Set})$ is the category whose objects are (set-indexed) families of sets $(X_i)_{i \in I}$, where a map $(X_i)_{i\in I} \to (Y_j)_{j \in J}$ consists of a map $u\colon I \to J$ and maps $X_i \to Y_{u(i)}$ for all $i \in I$. In one direction, a map $f\colon X \to I$ corresponds to the family of its fibers $(f^{-1}\{i\})_{i\in I}$. In the other direction, a family $(X_i)_{i\in I}$ corresponds to the map $\coprod_{i\in I} X_i \to I$. If we fix the target/indexing set I, we obtain an equivalence of categories

$$\mathrm{Set}_{/I} \simeq \mathrm{Set}^I.$$

The following proposition generalizes this fact to presheaves of sets (we recover the last equivalence by taking $\mathcal{C} = *$):

Proposition 2.46 (Slices of presheaf categories)**.** *Let $\mathcal{C}$ be a category and let $F \in \mathrm{P}(\mathcal{C})$ be a presheaf on $\mathcal{C}$. Then there is an equivalence of categories*

$$\mathrm{P}(\mathcal{C})_{/F} \underset{\mathrm{fib}_F}{\overset{\coprod_F}{\underset{\sim}{\leftrightarrows}}} \mathrm{P}(\mathrm{El}(F)),$$

described as follows. If $u\colon \mathrm{El}(F) \to \mathcal{C}$ is the forgetful functor, there is a tautological map $* \to u^*(F)$, whose adjoint $u_\sharp(*) \to F$ is an isomorphism.

- *Given $H \in \mathrm{P}(\mathrm{El}(F))$, the presheaf $\coprod_F H$ over F is $u_\sharp(H) \to u_\sharp(*) \simeq F$. Explicitly,*

$$\left(\coprod_F H\right)(X) = \coprod_{x\in F(X)} (H(X,x) \to *).$$

- *Given $G \in \mathrm{P}(\mathcal{C})_{/F}$, the presheaf $\mathrm{fib}_F(G)$ on $\mathrm{El}(F)$ is the pullback $u^*(G) \times_{u^*(F)} *$. Explicitly,*

$$\mathrm{fib}_F(G)(X,x) = G(X) \times_{F(X)} \{x\} \simeq \left\{ \begin{tikzcd} & G \arrow[d] \\ よ(X) \arrow[r, "x"] \arrow[ur, dashed] & F \end{tikzcd} \right\}.$$

Specializing to the case of a representable presheaf, we get:

Corollary 2.47. *Let $\mathcal{C}$ be a category, let $X \in \mathcal{C}$, and let $u\colon \mathcal{C}_{/X} \to \mathcal{C}$ be the forgetful functor. Then the functor $u_\sharp$ induces an equivalence of categories*

$$\mathrm{P}(\mathcal{C}_{/X}) \xrightarrow{\simeq} \mathrm{P}(\mathcal{C})_{/よ(X)}.$$

Specializing further to $\mathcal{C} = \mathrm{CAlg}^{\mathrm{op}}$, we get the following key result:

Corollary 2.48. *Let k be a ring and let $\varphi\colon \mathbb{Z} \to k$ be the unique map. Then the functor $\varphi_\sharp$ induces an equivalence of categories*

$$\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}) \xrightarrow{\simeq} \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/\operatorname{Spec}(k)}.$$

Remark 2.49. Because of Corollary 2.48, algebraic geometry over a base ring k is subsumed by algebraic geometry over $\mathbb{Z}$. In other words, working in the category $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$ of algebraic functors does not restrict the generality, and we will often do so from now on. Note that the functor Spec of Notation 2.7 is independent of k, in the sense that the following square commutes:

$$\begin{tikzcd} \mathrm{CAlg}_k^{\mathrm{op}} \arrow[r, "\sim"] \arrow[d, "\mathrm{Spec}"'] & (\mathrm{CAlg}^{\mathrm{op}})_{/k} \arrow[d, "\mathrm{Spec}"] \\ \mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set}) \arrow[r, "\simeq"] & \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/\operatorname{Spec}(k)}. \end{tikzcd}$$

Remark 2.50. If $f\colon X' \to X$ is a morphism of algebraic functors, there is a triple of adjoint functors

$$\begin{tikzcd} \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/X'} \arrow[r, bend left, "f_\sharp"] \arrow[r, bend right, "f_*"'] & \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})_{/X}, \arrow[l, "f^*"'] \end{tikzcd}$$

where $f^*(Y) = Y \times_X X'$ and $f_\sharp(Y') = Y'$. Given Proposition 2.46, this can be seen by applying Theorem 2.39 to the functor $u\colon \mathrm{El}(X') \to \mathrm{El}(X)$. This recovers Corollary 2.42 when f is $\operatorname{Spec}(\varphi)\colon \operatorname{Spec}(k') \to \operatorname{Spec}(k)$.

2.6. **Functions.** Recall that the affine line $\mathbb{A}^1$ is the forgetful functor

$$\mathbb{A}^1 \colon \mathrm{CAlg} \to \mathrm{Set}, \quad R \mapsto R.$$

Tautologically, $\mathbb{A}^1$ has a structure of ring object in $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, given by the factorization

$$\begin{tikzcd} & \mathrm{CAlg} \arrow[d, "\mathrm{forget}"] \\ \mathrm{CAlg} \arrow[ur, dashed, "\mathrm{id}"] \arrow[r, "\mathbb{A}^1"'] & \mathrm{Set}. \end{tikzcd}$$

Recall also that $\mathbb{G}_m$ is the subfunctor of $\mathbb{A}^1$ given by $R \mapsto R^\times$, which has a structure of abelian group (Example 2.32).

Definition 2.51 (Function and nonvanishing function)**.** Let X be an algebraic functor.

(i) A *function* on X is a map $X \to \mathbb{A}^1$. We denote by

$$\mathcal{O}(X) = \mathrm{Map}(X, \mathbb{A}^1)$$

the set of functions on X. The ring structure on $\mathbb{A}^1$ induces a ring structure on $\mathcal{O}(X)$.

(ii) An *nonvanishing function* on X is a map $X \to \mathbb{G}_\mathrm{m}$. We denote by

$$\mathcal{O}^\times(X) = \mathrm{Map}(X, \mathbb{G}_\mathrm{m})$$

the set of nonvanishing functions on X. The abelian group structure on $\mathbb{G}_\mathrm{m}$ induces an abelian group structure on $\mathcal{O}^\times(X)$.

Remark 2.52.

(i) Since $\mathbb{G}_\mathrm{m} \subset \mathbb{A}^1$, we have $\mathcal{O}^\times(X) \subset \mathcal{O}(X)$.

(ii) By the Yoneda lemma, there is a canonical isomorphism $\mathcal{O}(\mathrm{Spec}(R)) \simeq R$, i.e., the ring of functions on $\mathrm{Spec}(R)$ is R itself. Hence, when restricted to affine schemes, the functor $\mathcal{O} \colon \mathrm{Aff}^\mathrm{op} \to \mathrm{CAlg}$ is an equivalence of categories, which is inverse to Spec. Similarly, $\mathcal{O}^\times(\mathrm{Spec}(R)) \simeq R^\times$.

(iii) For a general algebraic functor X, we have

$$\mathcal{O}(X) \simeq \lim_{x \colon \mathrm{Spec}(R) \to X} R, \quad f \mapsto (f \circ x)_x,$$

where the limit is indexed by the category of elements $\mathrm{El}(X)^\mathrm{op}$ (Corollary 2.11). Since the unit group functor $R \mapsto R^\times$ preserves limits, it follows that $\mathcal{O}^\times(X)$ is precisely the unit group $\mathcal{O}(X)^\times$.

(iv) By definition, the functor

$$\mathcal{O} \colon \mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})^\mathrm{op} \to \mathrm{CAlg}, \quad X \mapsto \mathcal{O}(X),$$

is limit-preserving. By (ii) and the universal property of presheaves, it is the unique limit-preserving extension of the identity $\mathrm{CAlg} \to \mathrm{CAlg}$. Similarly, $\mathcal{O}^\times$ is the unique limit-preserving extension of the functor $\mathrm{CAlg} \to \mathrm{Ab}$, $R \mapsto R^\times$.

Remark 2.53 (Size issues)**.** The statement of Remark 2.52(iv) is actually nonsensical due to "size issues". Since the category CAlg is large, the limit in the formula for $\mathcal{O}(X)$ is indexed by a large category, so that the ring $\mathcal{O}(X)$ is sometimes large. In this case, $\mathcal{O}(X)$ is not an object of CAlg, which is the category of small rings. There are two standard ways to rectify this issue:

(i) One can simply replace the target of $\mathcal{O}$ by the category $\widehat{\mathrm{CAlg}}$ of large rings. One may then also replace the category of sets in the source by the category $\widehat{\mathrm{Set}}$ of large sets. We obtain the functor

$$\mathcal{O} \colon \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathrm{Set}})^\mathrm{op} \to \widehat{\mathrm{CAlg}},$$

which is the unique extension of the embedding $\mathrm{CAlg} \hookrightarrow \widehat{\mathrm{CAlg}}$ that preserves *large* limits.

(ii) One can replace the source of $\mathcal{O}$ by the subcategory $\mathrm{Fun}^\mathrm{acc}(\mathrm{CAlg}, \mathrm{Set})$ of *accessible* functors, which are the functors that are *small* colimits of representables. For an accessible functor X, $\mathcal{O}(X)$ is a small limit of small rings and hence is small. We therefore have a functor

$$\mathcal{O} \colon \mathrm{Fun}^\mathrm{acc}(\mathrm{CAlg}, \mathrm{Set})^\mathrm{op} \to \mathrm{CAlg},$$

which is the unique extension of the identity $\mathrm{CAlg} \to \mathrm{CAlg}$ that preserves *small* limits. All algebraic functors that arise in practice (including all schemes) are accessible, and all relevant constructions preserve accessibility, so that this restriction does not have any undesirable consequences.

2.7. **Closed and open subfunctors.** Roughly speaking, a *closed* subfunctor of an algebraic functor is a subfunctor defined by the vanishing of functions, while an *open* subfunctor is one defined by the nonvanishing of functions. This terminology is borrowed from topology, where the vanishing locus of a continuous function is closed and its nonvanishing locus is open. We will see later that there is in fact a topological interpretation of open subfunctors, though not of closed subfunctors.

Definition 2.54 (Vanishing and nonvanishing loci)**.** Let X be an algebraic functor and $F \subset \mathcal{O}(X)$ a set of functions on X.

(i) The *vanishing locus* of F is the subfunctor $\mathrm{V}(F) \subset X$ given by
$$\mathrm{V}(F)(R) = \{x \in X(R) \mid f(x) = 0 \text{ for all } f \in F\}.$$
(ii) The *nonvanishing locus* of F is the subfunctor of $\mathrm{D}(F) \subset X$ given by
$$\mathrm{D}(F)(R) = \{x \in X(R) \mid (f(x))_{f\in F} \text{ generates the unit ideal in } R\}.$$

Remark 2.55.

(i) It is clear that $\mathrm{V}(F)$ depends only on the ideal (F) generated by F, and $\mathrm{D}(F)$ only on the *radical* ideal $\sqrt{(F)}$ generated by F, since an ideal is the unit ideal if and only its radical is.
(ii) We have the following implications:
$$(F) = \mathcal{O}(X) \quad\Longrightarrow\quad \mathrm{V}(F) = \varnothing \quad\Longleftrightarrow\quad \mathrm{D}(F) = X.$$
Here, $\varnothing$ is the empty scheme from Example 2.30. The reverse implication holds if X is affine, by Proposition 2.65 below.

Example 2.56 (Punctured affine spaces)**.** Let I be a set. The *punctured affine I-space* $\mathbb{A}^I - 0$ is the nonvanishing locus of the coordinate functions $\{x_i \mid i \in I\}$ on $\mathbb{A}^I = \mathrm{Spec}(\mathbb{Z}[x_i \mid i \in I])$. Explicitly:
$$(\mathbb{A}^I - 0)(R) = \{a \in R^I \mid (a) = R\}.$$
Note that $\mathbb{A}^1 - 0$ is another name for the subfunctor $\mathbb{G}_\mathrm{m} \subset \mathbb{A}^1$. An I-tuple in R generating the unit ideal is also called a *unimodular row* of length I.

Remark 2.57. A set of functions $F \subset \mathcal{O}(X)$ induces a map $f\colon X \to \mathbb{A}^F$. By definition, we have
$$\mathrm{V}(F) = f^{-1}(0) \quad\text{and}\quad \mathrm{D}(F) = f^{-1}(\mathbb{A}^F - 0).$$

Remark 2.58. The terminology suggests that $\mathrm{D}(F)$ should in some sense be the complement of $\mathrm{V}(F)$ in X. This is true when evaluated on fields, but it is not true in the category of algebraic functors. In fact, they are not even disjoint: the intersection $\mathrm{V}(F) \cap \mathrm{D}(F)$ is the empty scheme (Example 2.30), which is not the initial object of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$. We will see later that $\mathrm{D}(F)$ is the complement of $\mathrm{V}(F)$ (i.e., the largest disjoint subobject) in various subcategories of $\mathrm{Fun}(\mathrm{CAlg}, \mathrm{Set})$, such as the category of schemes. Even then, the converse fails: $\mathrm{V}(F)$ cannot be the complement of $\mathrm{D}(F)$ in general, since it can happen that $\mathrm{V}(F) \neq \mathrm{V}(F')$ while $\mathrm{D}(F) = \mathrm{D}(F')$.

Proposition 2.59 (Formal properties of V and D)**.** *Let X be an algebraic functor.*

(i) *For any family $(F_i)_{i\in I}$ of subsets of $\mathcal{O}(X)$,*
$$\textstyle\bigcap_{i\in I} \mathrm{V}(F_i) = \mathrm{V}\big(\bigcup_{i\in I} F_i\big),$$
$$\textstyle\bigcup_{i\in I} \mathrm{D}(F_i) \subset \mathrm{D}\big(\bigcup_{i\in I} F_i\big),$$
and the inclusion is an equality on local rings.
(ii) *For any finite family $F_1, \dots, F_n$ of subsets of $\mathcal{O}(X)$,*
$$\mathrm{D}(F_1) \cap \dots \cap \mathrm{D}(F_n) = \mathrm{D}(F_1 \dots F_n),$$
$$\mathrm{V}(F_1) \cup \dots \cup \mathrm{V}(F_n) \subset \mathrm{V}(F_1 \dots F_n),$$
and the inclusion is an equality on integral domains.

Example 2.60. Since 2 and 3 generate the unit ideal in $\mathbb{Z}$, we have $\mathrm{V}(2) \cap \mathrm{V}(3) = \mathrm{V}(1) = \varnothing$ as subfunctors of $\mathrm{Spec}(\mathbb{Z})$. On the other hand, $\mathrm{D}(2) \cup \mathrm{D}(3) \neq \mathrm{D}(1) = \mathrm{Spec}(\mathbb{Z})$. For example, $\mathrm{id}_\mathbb{Z} \in \mathrm{Spec}(\mathbb{Z})(\mathbb{Z})$ belongs neither to $\mathrm{D}(2)(\mathbb{Z}) = \varnothing$ nor to $\mathrm{D}(3)(\mathbb{Z}) = \varnothing$.

Proposition 2.61 (Affineness of vanishing and nonvanishing loci)**.** *Let A be a ring.*

(i) *For any subset $F \subset A$, the quotient map $A \twoheadrightarrow A/(F)$ induces an isomorphism* $\mathrm{Spec}(A/(F)) \xrightarrow{\sim} \mathrm{V}(F) \subset \mathrm{Spec}(A)$. *In particular,* $\mathrm{V}(F)$ *is affine.*
(ii) *For any $f \in A$, the localization map $A \to A_f$ induces an isomorphism* $\mathrm{Spec}(A_f) \xrightarrow{\sim} \mathrm{D}(f) \subset \mathrm{Spec}(A)$. *In particular,* $\mathrm{D}(f)$ *is affine.*

If $F \subset A$ has more than one element, $\mathrm{D}(F) \subset \mathrm{Spec}(A)$ is usually not an affine scheme. For example, $\mathbb{A}^n - 0$ is not affine for $n \geq 2$ (see Example 2.74). This motivates the following definition:

Definition 2.62 (Quasi-affine scheme)**.** Let k be a ring. An algebraic k-functor X is a *quasi-affine k-scheme* if there exists a k-algebra A and a *finite* subset $F \subset A$ such that $X \simeq \mathrm{D}(F) \subset \mathrm{Spec}(A)$. A *quasi-affine scheme* is a quasi-affine $\mathbb{Z}$-scheme.

Definition 2.63 (Closed and open subfunctors)**.** Let X be an algebraic functor.

(i) A subfunctor $Z \subset X$ is *closed* if, for every $x\colon \mathrm{Spec}(R) \to X$, $x^{-1}(Z) = \mathrm{V}(F)$ for some $F \subset R$.
(ii) A subfunctor $U \subset X$ is *open* if, for every $x\colon \mathrm{Spec}(R) \to X$, $x^{-1}(U) = \mathrm{D}(F)$ for some $F \subset R$.

Remark 2.64. Vanishing loci are always closed subfunctors and nonvanishing loci are always open subfunctors, but the converse does not hold.

The following result generalizes the functorial Nullstellensatz (Corollary 2.25):

Proposition 2.65 (Classification of closed and open subfunctors of affine schemes)**.** *Let A be a ring.*

(i) *The construction $F \mapsto \mathrm{V}(F)$ induces an order-reversing bijection*

$$\{\textit{ideals in } A\} \xrightarrow{\sim} \{\textit{closed subfunctors of } \mathrm{Spec}(A)\}.$$

(ii) *The construction $F \mapsto \mathrm{D}(F)$ induces an order-preserving bijection*

$$\{\textit{radical ideals in } A\} \xrightarrow{\sim} \{\textit{open subfunctors of } \mathrm{Spec}(A)\}.$$

Definition 2.66 (Closed and open immersions)**.** Let $f\colon Y \to X$ be a map of algebraic functors.

(i) f is a *closed immersion* or *closed embedding* if it is a monomorphism whose image is a closed subfunctor of X.
(ii) f is an *open immersion* or *open embedding* if it is a monomorphism whose image is an open subfunctor of X.

Definition 2.67 (Locally closed subfunctor, immersion)**.** Let X be an algebraic functor.

(i) A subfunctor $Y \subset X$ is *locally closed* if there exists an open subfunctor $U \subset X$ containing Y as a closed subfunctor.
(ii) A map $f\colon Y \to X$ is an *immersion* if it is a monomorphism whose image is a locally closed subfunctor of X.

Proposition 2.68 (Closure properties of immersions)**.**

(i) *Consider a commutative triangle of algebraic functors*

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & X. & \end{tikzcd}$$

If f and g are closed immersions, so is h. If h is a closed immersion and f is a monomorphism, then g is a closed immersion. The same holds for open immersions and for immersions.

(ii) *Consider a cartesian square of algebraic functors*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

If f is a closed immersion, so is f'. The same holds for open immersions and for immersions.

2.8. **Zariski descent.** Let $(f_i)_{i\in I}$ be a family of element in a ring R that generates the unit ideal. Then the intersection of the vanishing loci $\mathrm{V}(f_i)$ is the empty scheme but, in general, it is not true that $\mathrm{Spec}(R)$ is the union of the nonvanishing loci $\mathrm{D}(f_i)$ (see Example 2.60). In this section, we will show that this becomes true if we compute the union in the category of affine schemes. Concretely, this means that a map $\mathrm{Spec}(R) \to \mathrm{Spec}(S)$ is uniquely determined by a family of maps $\mathrm{D}(f_i) \to \mathrm{Spec}(S)$ that agree on all the intersections $\mathrm{D}(f_i) \cap \mathrm{D}(f_j)$. This is one of the most important results in the foundations of algebraic geometry, which we will later recast as the statement that *affine schemes satisfy Zariski descent.*

Theorem 2.69 (Zariski descent for modules). *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R generating the unit ideal.*

(i) (Descent for morphisms) *For any R-modules M and N, the diagram*

$$\mathrm{Map}(N,M)\to\prod_{i\in I}\mathrm{Map}(N_{f_i},M_{f_i})\rightrightarrows\prod_{i,j\in I}\mathrm{Map}(N_{f_if_j},M_{f_if_j})$$

is an equalizer.

(ii) (Descent for objects) *Suppose given*
- *an R_{f_i}-module M_i for each $i\in I$ and*
- *an $R_{f_if_j}$-linear isomorphism $\alpha_{ij}\colon (M_i)_{f_j}\xrightarrow{\sim}(M_j)_{f_i}$ for each $(i,j)\in I^2$,*
- *such that $\alpha_{jk}\circ\alpha_{ij}=\alpha_{ik}\colon (M_i)_{f_jf_k}\xrightarrow{\sim}(M_k)_{f_if_j}$ for each $(i,j,k)\in I^3$.*

Then there exists an R-module M with R_{f_i}-linear isomorphisms $\beta_i\colon M_{f_i}\xrightarrow{\sim}M_i$ for all $i\in I$, such that $\alpha_{ij}\circ\beta_i=\beta_j\colon M_{f_if_j}\xrightarrow{\sim}(M_j)_{f_i}$ for all $(i,j)\in I^2$. Moreover, this data is unique up to unique isomorphism.

Taking $N=R$ in Theorem 2.69(i), we get the following special case:

Corollary 2.70. *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R generating the unit ideal. For any R-module M, the diagram*

$$M\to\prod_{i\in I}M_{f_i}\rightrightarrows\prod_{i,j\in I}M_{f_if_j}$$

is an equalizer in Mod_R.

Specializing further to $M=R$, we get the following result:

Corollary 2.71 (Zariski descent for affine schemes). *Let R be a ring and let $(f_i)_{i\in I}$ be a family of elements of R generating the unit ideal. For any affine scheme X, the diagram*

$$X(R)\to\prod_{i\in I}X(R_{f_i})\rightrightarrows\prod_{i,j\in I}X(R_{f_if_j})$$

is an equalizer.

Remark 2.72. Since $X(R)\simeq\mathrm{Map}(\mathrm{Spec}(R),X)$, $\mathrm{Spec}(R_{f_i})\simeq\mathrm{D}(f_i)$, and $\mathrm{Spec}(R_{f_if_j})\simeq\mathrm{D}(f_i)\cap\mathrm{D}(f_j)$, Corollary 2.71 says that $\mathrm{Spec}(R)$ is the union of the open subschemes $\mathrm{D}(f_i)$ in Aff. More precisely, if $\mathrm{Glue}(I)$ is the poset with morphisms $i\leftarrow(i,j)\to j$ for all $i,j\in I$, then $\mathrm{Spec}(R)$ is the colimit of the diagram $\mathrm{Glue}(I)\to\mathrm{Aff}$ sending i to $\mathrm{D}(f_i)$ and (i,j) to $\mathrm{D}(f_i)\cap\mathrm{D}(f_j)$.

The following result is a generalization of Corollary 2.71, of which it is in fact a formal consequence:

Corollary 2.73 (Functions on nonvanishing loci). *Let R be a ring and $(f_i)_{i\in I}$ a family of elements of R with image $F\subset R$. For any affine scheme X, there is an equalizer diagram*

$$\mathrm{Map}(\mathrm{D}(F),X)\to\prod_{i\in I}X(R_{f_i})\rightrightarrows\prod_{i,j\in I}X(R_{f_if_j}).$$

Example 2.74. Using Corollary 2.73 with $X=\mathbb{A}^1$, we can easily compute that the inclusion $\mathbb{A}^I-0\hookrightarrow\mathbb{A}^I$ induces an isomorphism $\mathcal{O}(\mathbb{A}^I)\xrightarrow{\sim}\mathcal{O}(\mathbb{A}^I-0)$ as soon as $|I|\geq 2$. Since $\mathcal{O}\colon\mathrm{Aff}^{\mathrm{op}}\to\mathrm{CAlg}$ is an equivalence of categories, this implies that $\mathbb{A}^I-0$ is not an affine scheme. In particular, if $n\geq 2$, $\mathbb{A}^n-0$ is an example of a quasi-affine scheme that is not affine.

Definition 2.75 (Zariski-local property). A property P of modules (resp. of linear maps, of algebras, etc.) is *Zariski-local* if, for any ring R and family $(f_i)_{i\in I}$ generating the unit ideal in R, an R-module M (resp. an R-linear map $M\to N$, an R-algebra A, etc.) has property P if and only if, for each $i\in I$, the R_{f_i}-module M_{f_i} (resp. the R_{f_i}-linear map $M_{f_i}\to N_{f_i}$, the R_{f_i}-algebra A_{f_i}, etc.) has property P.

Proposition 2.76 (Examples of Zariski-local properties). *The following properties of modules are Zariski-local:*

(i) *being zero,*
(ii) *finite generation,*
(iii) *finite presentation,*
(iv) *projectivity,*
(v) *flatness.*

The following properties of linear maps are Zariski-local:

(vi) *being zero,*
(vii) *injectivity,*
(viii) *surjectivity,*
(ix) *bijectivity.*

The following properties of algebras are Zariski-local:

(x) *finite generation,*
(xi) *finite presentation.*

The following properties of sequences of modules are Zariski-local:

(xii) *exactness.*

Remark 2.77. Further Zariski-local properties of modules are: being torsion, torsion-freeness. The following properties of modules are *not* Zariski-local: freeness, injectivity.

2.9. **Finiteness properties.** Recall that a k-algebra is *of finite presentation* if it is isomorphic to $k[\Sigma]$ where Σ is a system of finitely many polynomial equations in finitely many variables, and it is *of finite type* if it is isomorphic to $k[\Sigma]$ where Σ has finitely many variables (but any number of equations). We denote the respective full subcategories of CAlg_k by $\mathrm{CAlg}_k^{\mathrm{fp}}$ and $\mathrm{CAlg}_k^{\mathrm{ft}}$. It turns out that these finiteness conditions can naturally be expressed in terms of the algebraic k-functor $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$.

Definition 2.78 (Locally of finite presentation/type)**.** Let $X\colon \mathrm{CAlg}_k \to \mathrm{Set}$ be an algebraic k-functor.

(i) X is *locally of finite presentation* if it preserves filtered colimits.
(ii) X is *locally of finite type* if it preserves the colimits of filtered diagrams with injective transition maps.

Proposition 2.79. *Let k be a ring and A a k-algebra.*

(i) *A is of finite presentation if and only if* $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ *is locally of finite presentation.*
(ii) *A is of finite type if and only if* $\mathrm{Spec}(A)\colon \mathrm{CAlg}_k \to \mathrm{Set}$ *is locally of finite type.*

Example 2.80.

(i) $\mathbb{A}_k^I$ is locally of finite type if and only if I is a finite set, in which case it is also locally of finite presentation. The same holds for the punctured affine spaces $\mathbb{A}_k^I - 0$.
(ii) The affine k-scheme $\mathbb{A}(M)$ is locally of finite presentation (resp. of finite type) if and only if the k-module M is of finite presentation (resp. of finite type).
(iii) The algebraic k-functor $\mathbb{A}^\vee(M)$ of Remark 2.38 is locally of finite presentation for any k-module M, since the tensor product preserves colimits in each variable.
(iv) The affine $\mathbb{Z}$-schemes $*$, $\varnothing$, Idem, $\mathbb{G}_\mathrm{m}$, $\mathbb{G}_\mathrm{a}$, Mat_n, GL_n, and SL_n from §2.4 are all locally of finite presentation.

Remark 2.81. Since filtered colimits commute with finite limits in the category of sets, the condition of being locally of finite presentation or of finite type is preserved by finite limits in $\mathrm{Fun}(\mathrm{CAlg}_k, \mathrm{Set})$.

Remark 2.82 (Compatibility with base change)**.** Let $\varphi\colon k \to k'$ be a ring map. Since both the forgetful functor $\mathrm{CAlg}_{k'} \to \mathrm{CAlg}_k$ and its left adjoint $\varphi^*\colon \mathrm{CAlg}_k \to \mathrm{CAlg}_{k'}$ preserve filtered colimits, it follows from Corollary 2.42 that both base change along φ and Weil restriction along φ preserve the property of being locally of finite presentation or locally of finite type. On the other hand, the third functor $\varphi_\sharp$ usually does not. For example, $\mathrm{Spec}(\mathbb{C})$ is locally of finite presentation as an affine $\mathbb{R}$-scheme, but not as an affine $\mathbb{Q}$-scheme.

Recall that an algebraic k-functor can be thought of as a map of algebraic functors $X \to \mathrm{Spec}(k)$ (Corollary 2.48). Remark 2.82 implies that, for any cartesian square

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(k') \arrow[r] & \mathrm{Spec}(k), \end{tikzcd}$$

if f is locally of finite presentation or of finite type, so is f'. Consequently, we can extend Definition 2.78 to arbitrary maps of algebraic functors as follows:

Definition 2.83 (Morphism locally of finite presentation/type)**.** Let $f\colon X \to S$ be a map of algebraic functors.

(i) f is *locally of finite presentation* if, for every ring k and every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite presentation.

(ii) f is *locally of finite type* if, for every ring k and every k-point $\mathrm{Spec}(k) \to S$, the algebraic k-functor $X \times_S \mathrm{Spec}(k)$ is locally of finite type.

Concretely, under the equivalence of Corollary 2.48, the base change of $f\colon X \to S$ along $s\colon \mathrm{Spec}(k) \to S$ is the algebraic k-functor $\mathrm{CAlg}_k \to \mathrm{Set}$ given by

$$(\varphi\colon k \to R) \mapsto \{x \in X(R) \mid f(x) = s(\varphi) \text{ in } S(R)\} = \left\{ \begin{tikzcd} \mathrm{Spec}(R) \arrow[r, dashed, "x"] \arrow[d, "\mathrm{Spec}(\varphi)"'] & X \arrow[d, "f"] \\ \mathrm{Spec}(k) \arrow[r, "s"] & S \end{tikzcd} \right\}.$$

Example 2.84. Any open immersion is locally of finite presentation, and any closed immersion is locally of finite type. A closed immersion $i\colon Z \hookrightarrow X$ is locally of finite presentation if and only if, for every ring R and every $x\colon \mathrm{Spec}(R) \to X$, there exists a *finite* subset $F \subset R$ such that $x^{-1}(i(Z)) = \mathrm{V}(F)$.

Proposition 2.85 (Closure properties of morphisms locally of finite presentation/type)**.**

(i) *Consider a commutative triangle of algebraic functors*

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "h"'] & & Y \arrow[dl, "f"] \\ & X. & \end{tikzcd}$$

If f is locally of finite presentation, then g is locally of finite presentation if and only if h is. The same holds for "locally of finite type".

(ii) *Consider a cartesian square of algebraic functors*

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d] & X' \arrow[d] \\ Y \arrow[r, "f"] & X. \end{tikzcd}$$

If f is locally of finite presentation, so is f'. The same holds for "locally of finite type".

2.10. **The Nullstellensatz.** Consider a monic polynomial $f \in k[x]$ over a field k. By the functorial Nullstellensatz (Corollary 2.25), we know that f is determined by its zero sets in all k-algebras. On the other hand, by the elementary theory of fields, we know that f splits into linear factors over some finite field extension of k. Hence, if we know the zero sets of f over any finite field extension of k, then we know the original polynomial f provided it is separable (i.e., does not have multiple roots). In general, the zero sets of f over finite field extensions of k determine the *radical* of f, which is the product of the prime factors of f without multiplicity. The *Nullstellensatz* of Hilbert generalizes the latter statement to systems of polynomial equations in several variables: given $f_1, \dots, f_m \in k[x_1, \dots, x_n]$, the sets of common zeros of these polynomials in all finite field extensions of k determine the radical ideal $\sqrt{(f_1, \dots, f_n)}$. This nontrivial theorem was at the heart of classical algebraic geometry, which was only concerned with solutions of polynomial equations in fields. In this section, we review this result while also pointing out some shortcomings of the classical perspective.

For a ring k, define the maps

$$\{\text{subsets of } k^n\} \underset{\mathrm{V}}{\overset{\mathrm{I}}{\rightleftarrows}} \{\text{subsets of } k[x_1, \dots, x_n]\},$$

as follows:

$$\mathrm{I}(X) = \{f \in k[x_1, \dots, x_n] \mid f(x) = 0 \text{ for all } x \in X\},$$
$$\mathrm{V}(F) = \{x \in k^n \mid f(x) = 0 \text{ for all } f \in F\}.$$

Note that both maps are order-reversing and that $F \subset \mathrm{I}(\mathrm{V}(F))$ and $X \subset \mathrm{V}(\mathrm{I}(X))$ (in other words, this is an adjunction between posets). Note also that $\mathrm{I}(X)$ is always an ideal in $k[x_1, \dots, x_n]$, and it is even a radical ideal if k is reduced (if a power of f vanishes on X, so does f). Call a subset $X \subset k^n$ *algebraic* if it lies in the image of V, or equivalently if $X = \mathrm{V}(\mathrm{I}(X))$.

Theorem 2.86 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and let $n \in \mathbb{N}$. For any subset $F \subset k[x_1, \dots, x_n]$, we have*

$$\mathrm{I}(\mathrm{V}(F)) = \sqrt{(F)}.$$

Consequently, the maps I *and* V *define a one-to-one correspondence*

$$\{\textit{algebraic subsets of } k^n\} \underset{\mathrm{V}}{\overset{\mathrm{I}}{\rightleftarrows}} \{\textit{radical ideals in } k[x_1,\dots,x_n]\}.$$

We can upgrade this result to an equivalence of categories as follows. Define the category AffSet_k of *affine algebraic sets* over k as follows:

- An object of AffSet_k is a pair (n, X) with $n \geq 0$ and $X \subset k^n$ an algebraic subset.
- A morphism $(n, X) \to (m, Y)$ is a map $f \colon X \to Y$ such that there exists a polynomial map $k^n \to k^m$ extending f.

Recall that a ring R is *reduced* if 0 is the only nilpotent element of R. We denote by $\mathrm{CAlg}_k^{\mathrm{red}}$ the category of reduced k-algebras.

Corollary 2.87. *Let k be an algebraically closed field. Then there is an equivalence of categories*

$$\mathrm{AffSet}_k \xrightarrow{\sim} (\mathrm{CAlg}_k^{\mathrm{ft,red}})^{\mathrm{op}},$$
$$(n, X) \mapsto k[x_1,\dots,x_n]/\mathrm{I}(X).$$

Hence, AffSet_k *is equivalent to the full subcategory of* Aff_k *spanned by the reduced affine k-schemes of finite type.*

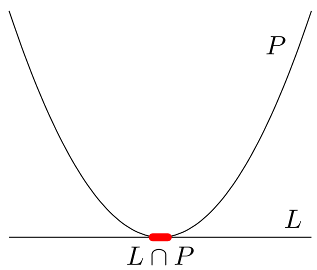
Remark 2.88. By Hilbert's Basissatz, "finite type" and "finite presentation" are equivalent for algebras over a field (and more generally over a noetherian ring).

We can also formulate a Nullstellensatz for an arbitrary field k as follows. Denote by $\mathrm{Field}_k^{\mathrm{fin}} \subset \mathrm{CAlg}_k$ the full subcategory of finite field extensions of k.

Corollary 2.89. *Let k be a field and let $n \in \mathbb{N}$. Then there is a one-to-one correspondence*

$$\{\textit{vanishing loci in } \mathbb{A}^n \colon \mathrm{Field}_k^{\mathrm{fin}} \to \mathrm{Set}\} \underset{\mathrm{V}}{\overset{\mathrm{I}}{\rightleftarrows}} \{\textit{radical ideals in } k[x_1,\dots,x_n]\}.$$

Example 2.90 (Non-reduced intersections)**.** Even in the context of algebraic geometry over an algebraically closed field k, there are geometric phenomena that are not captured by only considering solutions in k. Consider for examples the vanishing loci $L = \mathrm{V}(y)$ and $P = \mathrm{V}(y - x^2)$ in $\mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$. Since the k-algebras $k[x,y]/(y) \simeq k[t]$ and $k[x,y]/(y-x^2) \simeq k[t]$ are reduced, these affine k-schemes are determined by the algebraic sets $L(k)$ and $P(k)$ in k^2 (by the Nullstellensatz). The intersection $L(k) \cap P(k)$ is the algebraic set $\{(0,0)\} \subset k^2$, which in turn corresponds to the subfunctor $\mathrm{V}(x,y) \subset \mathbb{A}^2_k$. However, the functorial intersection $L \cap P \subset \mathbb{A}^2_k$ is the subfunctor $\mathrm{V}(x^2, y)$, which is isomorphic to $\mathrm{Spec}(k[t]/(t^2))$. One can think of $\mathrm{V}(x^2,y)$ as a first-order infinitesimal neighborhood of the origin $\mathrm{V}(x,y)$ along the x-axis; this captures the fact that the line L is tangent (to first order) to the parabola P, so that they both contain the same infinitesimal horizontal segment at the origin. This residual tangency information in the intersection can only be seen by evaluating the functor $L \cap P$ on non-reduced k-algebras.



This also resolves another issue in classical algebraic geometry, which is that intersections do not vary nicely in families. Consider for example the family of horizontal line $L_a = \mathrm{V}(y - a)$ for $a \in k$. The intersection $L_a(k) \cap P(k)$ has exactly two points for any $a \neq 0$ (since k is algebraically closed), but only a single point when $a = 0$. On the other hand, the scheme-theoretic intersection $L_a \cap P$ is given by a 2-dimensional k-algebra for *all* $a \in k$, namely $k \times k$ when $a \neq 0$ and $k[t]/(t^2)$ when $a = 0$.

Example 2.91 (Geometry in mixed characteristic)**.** Another aspect that is not captured by classical algebraic geometry over fields is algebraic geometry in *mixed characteristic*, i.e., involving rings R that do not contain any field. This is especially relevant in number theory, which studies rings of integers in finite extensions of $\mathbb{Q}$. Such rings can map to fields with different characteristics, which sometimes allows us to transport results from one characteristic to another. As a very basic example, consider the following proof that $\sqrt{2}$ is irrational (which is a reformulation of the usual argument). A positive rational number x such that $x^2 = 2$ is the same thing as an element of $X(\mathbb{Z})$ where $X \subset \mathbb{P}^1$ is the solution functor to the homogeneous polynomial equation $x^2 = 2y^2$. Since X is a functor, the ring map $\mathbb{Z} \to \mathbb{Z}/4$ induces a map $X(\mathbb{Z}) \to X(\mathbb{Z}/4)$. Since the squares in $\mathbb{Z}/4$ are 0 and 1, none of the six elements of $\mathbb{P}^1(\mathbb{Z}/4)$ satisfy the equation $x^2 = 2y^2$, so that $X(\mathbb{Z}/4)$ is empty. It follows that $X(\mathbb{Z})$ is also empty, i.e., that there does not exist $x \in \mathbb{Q}$ with $x^2 = 2$.

3. Projective geometry

3.1. **Projective spaces over a field.** Let k be a field. The classical projective n-space over k is the set of lines through the origin in k^{n+1}:

$$\mathbb{P}^n(k) = \{1\text{-dimensional subspaces of } k^{n+1}\}.$$

Given a nonzero $(n+1)$-tuple $(a_0, \dots, a_n) \in k^{n+1} - \{0\}$, we denote by $[a_0 : \dots : a_n]$ the 1-dimensional subspace of k^{n+1} containing $(a_0, \dots, a_n)$. This identifies $\mathbb{P}^n(k)$ with the set of orbits of the (free) action of $k^\times$ on $k^{n+1} - \{0\}$ by scalar multiplication:

$$\begin{aligned} (k^{n+1} - \{0\})/k^\times &\xrightarrow{\sim} \mathbb{P}^n(k), \\ (a_0, \dots, a_n) &\mapsto [a_0 : \dots : a_n]. \end{aligned}$$

The set $\mathbb{P}^n(k)$ is the union of $n+1$ copies $U_0, \dots, U_n$ of $\mathbb{A}^n(k) = k^n$, where

$$U_i = \{[a_0 : \dots : a_n] \in \mathbb{P}^n(k) \mid a_i = 1\}.$$

The complement $H_i = \mathbb{P}^n(k) - U_i$ is given by

$$H_i = \{[a_0 : \dots : a_n] \in \mathbb{P}^n(k) \mid a_i = 0\}$$

and can be identified with $\mathbb{P}^{n-1}(k)$. We often think of $\mathbb{P}^n(k)$ as the completion of $U_0 = \mathbb{A}^n(k)$ obtained by adding a point "at infinity" in every radial direction. These points at infinity form the *hyperplane at infinity* $H_0 = \mathbb{P}^{n-1}(k)$ in $\mathbb{P}^n(k)$.

Since the k-vector space k^{n+1} is canonically self-dual, we can identify $\mathbb{P}^n(k)$ with the set of 1-dimensional *quotient spaces* of k^{n+1}:

$$\begin{aligned} \mathbb{P}^n(k) &\simeq \{1\text{-dimensional quotient spaces of } k^{n+1}\}, \\ L &\mapsto k^{n+1}/L^\perp. \end{aligned}$$

Concretely, the line $[a_0 : \dots : a_n]$ corresponds to the coimage of the map $(a_0, \dots, a_n) \colon k^{n+1} \twoheadrightarrow k$.