COMPACT MORPHISMS AND THE AXIOM (AB6)

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These notes were prepared for Marc Hoyois's seminar on Efimov K-theory. They are based on Efimov's paper [Efi24] and the lecture notes of Krause, Nikolaus, and Pützstück [KNP24].

For simplicity, we only deal with the case of presentable stable ∞-categories. For the case of general presentable ∞-categories, see [KNP24].

1. Compact morphisms

Let $\mathcal{C}$ be a presentable stable ∞-category.

Definition 1.1 (Compact morphism)**.** A morphism $f : X \to Y$ in $\mathcal{C}$ is called *compact* if for every filtered diagram $\{Z_i\}_{i\in I}$ in $\mathcal{C}$, and any morphism $Y \to \operatorname{colim}_{i\in I} Z_i$, the composition $X \to \operatorname{colim}_{i\in I} Z_i$ factors through some Z_i.

Example 1.2. For a morphism $f : X \to Y$ in $\mathcal{C}$, if it factors through a compact object K in $\mathcal{C}$, then f is compact. In fact, for any map $g : Y \to \operatorname{colim}_{i\in I} Z_i$, $K \to Y \to \operatorname{colim}_{i\in I} Z_i$ already factors through some Z_i, hence gf factors through some Z_i.

Conversely, for $\mathcal{C}$ compactly generated, if $f : X \to Y$ is compact, we can write Y as a filtered colimit of compact objects $Y \simeq \operatorname{colim}_{i\in I} Y_i$, then f factors through some Y_i.

Remark 1.3. An equivalent definition of a compact morphism is that $f : X \to Y$ is compact iff for any object Z in $\operatorname{Ind}(\mathcal{C})$, and any morphism $\mathcal{Y}(Y) \to \mathcal{Y}k(Z)$, there exists a morphism $\mathcal{Y}(X) \to Z$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{Y}(X) & \dashrightarrow & Z \\ {\scriptstyle \mathcal{Y}(f)}\downarrow & & \downarrow \\ \mathcal{Y}(Y) & \longrightarrow & \mathcal{Y}k(Z) \end{array}$$

where $\mathcal{Y} : \mathcal{C} \to \operatorname{Ind}(\mathcal{C})$ and $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$ are the canonical adjunction of Yoneda embedding and taking colimits.

Lemma 1.4. *(1) If $X \to Y$ is a compact morphism, then for any morphism $Y \to Z$ and any morphism $W \to X$, the composition $W \to X \to Y \to Z$ is a compact morphism.*

(2) Let $\mathcal{C}$ and $\mathcal{D}$ be presentable stable ∞-categories, and $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful colimit-preserving functor. Then F reflects compact morphisms.

Proof. (1) is clear from the definition. For (2), let $f : X \to Y$ be a morphism in $\mathcal{C}$, and suppose that $F(f) : F(X) \to F(Y)$ is compact in $\mathcal{D}$. For any filtered diagram $\{Z_i\}_{i\in I}$ in $\mathcal{C}$, and any morphism $g : Y \to \operatorname{colim}_{i\in I} Z_i$, the composition $F(gf)$:

Date: November 3, 2026.

$F(X) \to \operatorname{colim}_{i\in I} F(Z_i)$ factors through some $F(Z_i)$. Since F is fully faithful, we have that $g \circ f : X \to \operatorname{colim}_{i\in I} Z_i$ factors through Z_i, hence f is compact. □

Now we give equivalent characterizations of compact morphisms in dualizable stable ∞-categories.

Proposition 1.5 ([KNP24, Lemma 2.3.17])**.** *Let $\mathcal{C}$ be a dualizable stable ∞-category, and $f : X \to Y$ be a morphism in $\mathcal{C}$, then the following are equivalent:*

(1) f is a compact morphism.
(2) the morphism $\mathcal{Y}(f) : \mathcal{Y}(X) \to \mathcal{Y}(Y)$ factors through $\hat{\mathcal{Y}}(Y)$ in $\operatorname{Ind}(\mathcal{C})$, where $\hat{\mathcal{Y}} : \mathcal{C} \to \operatorname{Ind}(\mathcal{C})$ is the left adjoint of $k : \operatorname{Ind}(\mathcal{C}) \to \mathcal{C}$.
(3) the morphism $\hat{\mathcal{Y}}(f) : \hat{\mathcal{Y}}(X) \to \hat{\mathcal{Y}}(Y)$ factors through $\mathcal{Y}(X)$.
(4) the morphism $\hat{\mathcal{Y}}(f) : \hat{\mathcal{Y}}(X) \to \hat{\mathcal{Y}}(Y)$ is compact in $\operatorname{Ind}(\mathcal{C})$.

Proof. (1) ⇒ (2): Since $k\hat{\mathcal{Y}}(Y) \simeq Y$, this follows directly from Remark 1.3.

(2) ⇒ (3): Under the adjunction $\hat{\mathcal{Y}} \dashv k$, the diagram on the left corresponds to the diagram on the right:

$$\begin{array}{ccc} \hat{\mathcal{Y}}(X) & \longrightarrow & \hat{\mathcal{Y}}(Y) \\ \downarrow & \nearrow & \downarrow \\ \mathcal{Y}(X) & \longrightarrow & \mathcal{Y}(Y) \end{array} \qquad \begin{array}{ccc} X & \longrightarrow & k\hat{\mathcal{Y}}(Y) \\ \downarrow\wr & \nearrow & \downarrow\wr \\ k\mathcal{Y}(X) & \longrightarrow & k\mathcal{Y}(Y). \end{array}$$

Condition (2) gives a dashed arrow in the left diagram, making the lower triangle commute. To see the upper triangle commutes, it suffices to show the upper triangle in the right diagram commutes, which can be deduced from the commutativity of the lower triangle in the right and the commutativity of the outer square.

(3) ⇒ (4): Trivial. (4) ⇒ (1): By Lemma 1.4(2), $\hat{\mathcal{Y}}$ reflects compact morphisms, hence f is compact. □

Corollary 1.6. *In a dualizable category $\mathcal{C}$, any compact morphism can be factored as two compact morphisms.*

Proof. Let $f : X \to Y$ be a compact morphism. By Proposition 1.5, $\mathcal{Y}(X) \to \mathcal{Y}(Y)$ factors through $\hat{\mathcal{Y}}(Y)$, then let $\hat{\mathcal{Y}}(Y) = \operatorname{colim}_{i\in I} \mathcal{Y}(Z_i)$, then $Y \simeq \operatorname{colim}_{i\in I} Z_i$, the natural map $\hat{\mathcal{Y}} \Rightarrow \mathcal{Y}$ induces an equivalence $\operatorname{colim}_{i\in I} \hat{\mathcal{Y}}(Z_i) \simeq \operatorname{colim}_{i\in I} \mathcal{Y}(Z_i)$. Therefore, $\mathcal{Y}(X) \to \hat{\mathcal{Y}}(Y)$ factors as $\mathcal{Y}(X) \to \hat{\mathcal{Y}}(Z_i) \to \mathcal{Y}(Z_i) \to \hat{\mathcal{Y}}(Y)$, hence $X \to Z_i$ and $Z_i \to Y$ are compact. □

Lemma 1.7 ([Efi24, Proposition 2.41])**.** *Let $\mathcal{C}$ be a presentable stable ∞-category, then a morphism $f : X \to Y$ in $\mathcal{C}$ is compact if and only if for any $Z \in \operatorname{Ind}(\mathcal{C})$ such that $k(Z) \simeq 0$, the map $\pi_0(\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\mathcal{Y}(Y), Z)) \to \pi_0(\operatorname{Map}_{\operatorname{Ind}(\mathcal{C})}(\mathcal{Y}(X), Z))$ is null-homotopic.*

Proof. For the "only if" direction, let $f : X \to Y$ be a compact morphism, for any filtered diagram $\{z_i\}_{i\in I}$ in $\mathcal{C}$ such that $\operatorname{colim}_{i\in I} z_i \simeq 0$, and any morphism $Y \to z_{i_0}$ for some $i_0 \in I$, consider the map $\operatorname{colim}_{i\in I_{i_0/}} \ker(Y \to z_i) \to Y$, which is an equivalence, let g be the inverse of this equivalence, then by the compactness of f, the composition $gf : X \to Y \to \operatorname{colim}_{i\in I_{i_0/}} \ker(Y \to z_i)$ factors through some $\ker(Y \to z_i)$, hence the composition $X \to Y \to z_i$ is null-homotopic.

For the "if" direction, let $f : X \to Y$ satisfy the condition in the lemma, then for $Z \in \mathrm{Ind}(\mathcal{C})$, put $Q = \mathrm{cofib}(Z \to \mathcal{Y}k(Z))$ and consider the following commutative diagram of exact sequences:

$$\begin{array}{ccccc} \pi_0 \mathrm{Map}(\mathcal{Y}(Y), Z) & \to & \pi_0 \mathrm{Map}(\mathcal{Y}(Y), \mathcal{Y}k(Z)) & \to & \pi_0 \mathrm{Map}(\mathcal{Y}(Y), Q) \\ \downarrow & & \downarrow & & \downarrow \\ \pi_0 \mathrm{Map}(\mathcal{Y}(X), Z) & \to & \pi_0 \mathrm{Map}(\mathcal{Y}(X), \mathcal{Y}k(Z)) & \to & \pi_0 \mathrm{Map}(\mathcal{Y}(X), Q). \end{array}$$

The condition in the lemma implies that the right vertical map is null-homotopic, hence the middle vertical map maps any object to the image of the lower left horizontal map, which is equivalent to the compactness of f. □

Corollary 1.8. *An object X in $\mathcal{C}$ is compact if and only if the identity $X \to X$ is a compact morphism.*

Proof. By Lemma 1.7, the identity $X \to X$ is compact iff for any $Z \in \ker(\mathrm{Ind}(\mathcal{C}) \to \mathcal{C})$, $\pi_0(\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathcal{Y}(X), Z))$ is zero, since $\ker(\mathrm{Ind}(\mathcal{C}) \to \mathcal{C})$ is stable, this is equivalent to requiring that, for any $Z \in \mathrm{Ind}(\mathcal{C})$, $\mathrm{map}(\mathcal{Y}(X), Z) \to \mathrm{map}(\mathcal{Y}(X), \mathcal{Y}k(Z))$ is an equivalence. □

Next, we give a criterion for dualizability of a presentable stable category in terms of compact morphisms.

Definition 1.9. An object X in $\mathcal{C}$ is called *compactly exhaustive* if it can be written as a sequential colimit $X_0 \to X_1 \to X_2 \to \cdots$, such that each $X_i \to X_{i+1}$ is a compact morphism.

Lemma 1.10. *Let $X_0 \to X_1 \to X_2 \to \cdots$ be a sequence of compact morphisms in $\mathcal{C}$, consider the colimit $\hat{X} = \mathrm{colim}_i \mathcal{Y}(X_i)$ in $\mathrm{Ind}(\mathcal{C})$, then $\hat{X}$ is left orthogonal to $\ker(\mathrm{Ind}(\mathcal{C}) \to \mathcal{C})$. In particular, for any $Z \in \mathrm{Ind}(\mathcal{C})$, we have*

$$\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\hat{X}, Z) \simeq \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_i X_i, k(Z))$$

Proof. Let $W \in \ker(\mathrm{Ind}(\mathcal{C}) \to \mathcal{C})$, then by Lemma 1.7, for any i, the map

$$\pi_0(\mathrm{Map}(\mathcal{Y}(X_{i+1}), W)) \to \pi_0(\mathrm{Map}(\mathcal{Y}(X_i), W))$$

is zero, hence $\pi_0(\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(\mathrm{colim}_i \mathcal{Y}(X_i), W))$ is zero, so $\mathrm{map}(\hat{X}, W) \simeq 0$. In particular, for any $Z \in \mathrm{Ind}(\mathcal{C})$, we have $\mathrm{map}(\hat{X}, Z) \simeq \mathrm{map}(\hat{X}, \mathcal{Y}(k(Z)))$, the latter is equivalent to $\mathrm{map}(\mathrm{colim}_i X_i, k(Z))$ by adjunction. □

Corollary 1.11. *Compactly exhaustive objects are ω_1-compact in $\mathcal{C}$.*

Proposition 1.12. *Let $\mathcal{C}$ be a presentable stable ∞-category, the following are equivalent:*

(1) $\mathcal{C}$ is dualizable
(2) $\mathcal{C}$ is generated by compactly exhaustive objects.

Proof. (2) ⇒ (1): For $X \in \mathcal{C}$ compactly exhaustive, by Lemma 1.10, the functor $\mathrm{map}(X, k(-))$ is representable. Since $\mathcal{C}$ is generated by compactly exhaustive objects, the functor $\mathrm{map}(X, k(-))$ for general X can be written as a limit of representable functors, hence is representable.

(1) ⇒ (2): It suffices to prove that compactly exhaustive objects are jointly conservative. Since $\mathcal{C}$ is stable, it suffices to show that for any $Z \in \mathcal{C}$, if $\mathrm{map}(X, Z) \simeq 0$

for any compactly exhaustive object X, then $Z \simeq 0$. Now let $Z \in \mathcal{C}$ such that $\mathrm{map}(X, Z) \simeq 0$ for any compactly exhaustive object X, Assume that C is κ-compactly generated. By full faithfulness of $\hat{\mathcal{Y}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}^\kappa)$ and the compact generation of $\mathrm{Ind}(\mathcal{C}^\kappa)$, it is sufficient to show that $\pi_0(\mathrm{map}(\mathcal{Y}(X), \hat{\mathcal{Y}}(Z))) = 0$ for every $X \in \mathcal{C}^\kappa$.

Consider any map $\mathcal{Y}(X) \to \hat{\mathcal{Y}}(Z)$, let $X_0 = X$, by the argument in Corollary 1.6, we can find $X_1 \in \mathcal{C}^\kappa$ and a map $X_0 \to X_1$ such that $X_0 \to X_1$ is compact and $\mathcal{Y}(X) \to \hat{\mathcal{Y}}(Z)$ factors as $\mathcal{Y}(X_0) \to \mathcal{Y}(X_1) \to \hat{\mathcal{Y}}(Z)$, applying this procedure inductively, we get a sequence of compact morphisms $X_0 \to X_1 \to X_2 \to \cdots$ and a map $\mathrm{colim}_i \mathcal{Y}(X_i) \to \hat{\mathcal{Y}}(Z)$ lifting $\mathcal{Y}(X) \to \hat{\mathcal{Y}}(Z)$ By Lemma 1.10, $\mathrm{colim}_i \mathcal{Y}(X_i)$ is equivalent to $\hat{\mathcal{Y}}(\mathrm{colim}_i X_i)$, we have $X \to Z$ null-homotopic, $\pi_0(\mathrm{map}(\mathcal{Y}(X), \hat{\mathcal{Y}}(Z))) = 0$. □

Corollary 1.13. *Dualizable stable ∞-categories are ω_1-presentable, and the ω_1-compact objects are generated by compactly exhaustive objects.*

By Proposition 1.5, a morphism $X \to Y$ being compact is witnessed by a lift $\mathcal{Y}(X) \to \hat{\mathcal{Y}}(Y)$, so we can consider the compact assembled Hom space $\mathrm{Hom}(\mathcal{Y}(X), \hat{\mathcal{Y}}(Y))$. The following proposition describes the relationship between these two Hom spaces.

Proposition 1.14. *Let $\mathcal{C}$ be a dualizable stable ∞-category, and let $\mathcal{D}$ be an ∞-category admitting filtered colimits. Write $\mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D})$ for the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by the functors preserving filtered colimits. Then $\mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D})$ is a Bousfield colocalization of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, with right adjoint*

$$\mathrm{asm}_{\mathrm{filt}} : \mathrm{Fun}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D}), \qquad F \longmapsto k_{\mathcal{D}} \circ \mathrm{Ind}(F) \circ \hat{\mathcal{Y}},$$

where $k_{\mathcal{D}} : \mathrm{Ind}(\mathcal{D}) \to \mathcal{D}$ is the functor taking filtered colimits. In particular, for any functor $F : \mathcal{C} \to \mathcal{D}$, the counit

$$k_{\mathcal{D}} \circ \mathrm{Ind}(F) \circ \hat{\mathcal{Y}} \longrightarrow F$$

is the filtered colimit assembly map, i.e. a terminal object in $\mathrm{Fun}^{\mathrm{filt}}(\mathcal{C}, \mathcal{D})_{/F}$.

Proof. The adjunction $\hat{\mathcal{Y}} \dashv k$ induces an adjunction $k^* \dashv \hat{\mathcal{Y}}^*$ on $\mathrm{Fun}(\mathrm{Ind}(C), D)$ and $\mathrm{Fun}(C, D)$, since both functors preserve colimits, restriction on full subcategories gives $k^* \dashv \hat{\mathcal{Y}}^* : \mathrm{Fun}^{\mathrm{filt}}(C, D) \rightleftarrows \mathrm{Fun}^{\mathrm{filt}}(\mathrm{Ind}(C), D)$. Identifying the latter category with $\mathrm{Fun}(C, D)$ gives the desired result. □

Corollary 1.15. *For a dualizable stable ∞-category, $\mathrm{Hom}(\mathcal{Y}(X), \hat{\mathcal{Y}}(-))$ is the filtered colimit assembly map of $\mathrm{Hom}(\mathcal{Y}(X), \mathcal{Y}(-))$*

Proof. By Proposition 1.14, the filtered colimit assembly map of $\mathrm{Hom}_{\mathcal{C}}(X, -)$ is $k \circ \mathrm{Ind}(\mathrm{Hom}(X, -)) \circ \hat{\mathcal{Y}}$. The composition of the first two maps is the left Kan extension of $\mathrm{Hom}(X, -)$, hence is $\mathrm{Hom}(\mathcal{Y}(X), -)$. □

2. The axiom (AB6)

In this section, we discuss the axiom (AB6), which is an equivalent condition for a presentable stable ∞-category to be dualizable.

Let $\mathcal{C}$ be a presentable stable ∞-category.

Definition 2.1. We say that $\mathcal{C}$ satisfies the axiom (AB6) if filtered colimits distribute over infinite products in $\mathcal{C}$, i.e. for any I-indexed class of filtered diagrams $\{X_{i,j}\}_{j\in J_i}$ in $\mathcal{C}$, the canonical map

$$\operatorname*{colim}_{j\in\prod_{i\in I} J_i} \prod_{i\in I} X_{i,j(i)} \longrightarrow \prod_{i\in I} \operatorname*{colim}_{j\in J_i} X_{i,j}$$

is an equivalence.

Lemma 2.2. *(AB6) holds in the category of anima* Ani.

Proof. Let $\{X_{i,j}\}_{j\in J_i}$ be an I-indexed class of filtered diagrams in Ani, let $f : P \to Q$ be the canonical map in Definition 2.1, since (AB6) holds in the category of sets, the map $\pi_0(f)$ is an equivalence.

In each component of P, choose a base point p_0 represented by a tuple $(x_i)_{i\in I}$ at a stage $j_0 \in \prod_i J_i$. Put $K_i = (J_i)_{j_0(i)/}$. For $u_i \colon j_0(i) \to j_i$ in J_i and $n \geq 1$, set

$$A_i(u_i) = \pi_n\big(X_{i,j_i}, X_i(u_i)(x_i)\big).$$

Each K_i is filtered and $K_i \to J_i$ is cofinal. Likewise, $\prod_i K_i$ is the undercategory of j_0 in $\prod_i J_i$ and $\prod_i K_i \to \prod_i J_i$ is cofinal. Since π_n preserves products and filtered colimits of pointed anima, the map induced by f on π_n at p_0 identifies with

$$\operatorname*{colim}_{(u_i)\in\prod_i K_i} \prod_i A_i(u_i) \longrightarrow \prod_i \operatorname*{colim}_{u_i\in K_i} A_i(u_i).$$

By (AB6) for sets, this map is an isomorphism, hence $\pi_n(f, p_0)$ is an isomorphism for all $n \geq 1$ and $p_0 \in P$, so f is an equivalence. □

Corollary 2.3. *Any compactly generated ∞-category satisfies the axiom (AB6).*

Proof. Let $\mathcal{C}$ be a compactly generated ∞-category. Then compact objects in $\mathcal{C}$ are jointly conservative, and for compact objects X, $\mathrm{Hom}(X, -)$ commutes with filtered colimits and infinite products, then by Lemma 2.2, $\mathcal{C}$ satisfies (AB6). □

Proposition 2.4. *For a presentable stable ∞-category $\mathcal{C}$, the following are equivalent:*

(1) $\mathcal{C}$ is dualizable.
(2) $\mathcal{C}$ satisfies the axiom (AB6).

Proof. (1) ⇒ (2): Since the functor $k : \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ preserves limits and colimits, for any I-indexed class of filtered diagrams in $\mathcal{C}$, we can lift it to $\mathrm{Ind}(\mathcal{C})$, since (AB6) holds in $\mathrm{Ind}(\mathcal{C})$, passing to k gives the desired result.

(2) ⇒ (1): To show that $\mathcal{C}$ is dualizable, it suffices to show that $k : \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$ preserves infinite products for κ sufficiently large. For any I-indexed class of objects $Y_i$in $\mathrm{Ind}(\mathcal{C}^\kappa)$, we can write each Y_i as a filtered colimit of κ-compact objects, namely $\mathrm{colim}_{J_i} \mathcal{Y}(X_{i,j})$, then by (AB6) in $\mathrm{Ind}(\mathcal{C}^\kappa)$, we have

$$\prod_{i\in I} Y_i \simeq \operatorname*{colim}_{j\in\prod_{i\in I} J_i} \prod_{i\in I} \mathcal{Y}(X_{i,j(i)}) \simeq \operatorname*{colim}_{j\in\prod_{i\in I} J_i} \mathcal{Y}\Big(\prod_{i\in I} X_{i,j(i)}\Big)$$

passing to k we have

$$k\Big(\prod_{i\in I} Y_i\Big) \simeq \operatorname*{colim}_{j\in\prod_{i\in I} J_i} \prod_{i\in I} X_{i,j(i)}$$

unpacking the right hand side using (AB6) in $\mathcal{C}$, we get $\prod_{i\in I} \mathrm{colim}_{j\in J_i} X_{i,j} \simeq \prod_{i\in I} k(Y_i)$, hence k preserves infinite products. □

References

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