CALKIN CATEGORIES AND COLIMITS OF DUALIZABLE CATEGORIES

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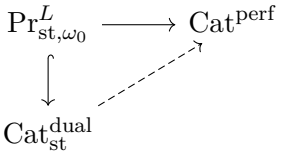
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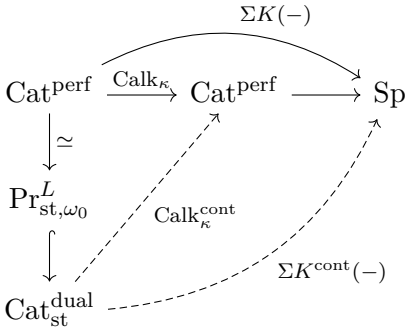
1. Introduction

A naive way to extend K-theory, or any localizing invariant, would be to have a retract of the inclusion $\mathrm{Pr}^L_{\mathrm{st},\omega_0} \hookrightarrow \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$



Here the functor $(-)^{\omega_0}$ provides such a retraction, but it fails to give meaningful K-theory: nontrivial dualizable categories need not have any nontrivial compact objects, as shown in Keller's example.

A pragmatic solution is given by finding an extension of the classical Calkin category functor, where the Calkin category functor induces the suspension functor on K-theory.



Now any dualizable category $\mathcal{C}$ fits in a short exact sequence $\mathcal{C} \to \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ for some $\mathcal{A}, \mathcal{B} \in \mathrm{Cat}^{\mathrm{perf}}$. So then we want $K^{\mathrm{cont}}(\mathcal{C})$ to be the fiber of $K(\mathcal{A}) \to K(\mathcal{B})$. We show that there exists well-defined and functorial choice for $\mathcal{A}, \mathcal{B}$. This also mirrors and extends the classical Calkin category, as the chosen $\mathcal{A}$ has countably infinite coproducts. Furthermore, we prove that this construction preserves short exact sequences in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$.

In the second half, we prove that colimits in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ are computed in $\mathrm{Pr}^L_{\mathrm{st}}$ and that filtered colimits in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ preserve fully faithful functors and short exact sequences.

1.1. **Conventions.** Throughout this paper, κ is always be an uncountable regular cardinal. Furthermore, categories $\mathcal{C}, \mathcal{D}$ are always assumed to be at least presentable and stable, and a functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ is assumed to be a left adjoint, unless stated otherwise. We assume that dualizable categories are always stable. We denote by $\mathcal{F}^R$ the corresponding right adjoint of a left adjoint $\mathcal{F}$. We denote the Yoneda embedding by $\yoneda$, and $\widehat{\yoneda}$ is the left adjoint of the formal colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.

The numbering of this paper refers to the fourth version of [Efi26].

2. Continuous Calkin Category

Definition 2.1 (Classical Calkin Category)**.** The Calkin category functor $\mathrm{Calk}_\kappa(-)\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{Cat}^{\mathrm{perf}}$ is defined as the idempotent completion of the Verdier quotient $(\mathrm{Ind}(-))^\kappa/\yoneda(-)$.

Remark 2.2. The category $\mathrm{Ind}(\mathcal{A})^\kappa$ admits countably infinite coproducts, since κ is uncountable. Hence by the Eilenberg swindle, we have a natural equivalence $K(\mathrm{Calk}_\kappa(\mathcal{A})) \simeq \Sigma K(\mathcal{A})$.

We have seen in the second talk that for a compactly assembled category $\mathcal{C}$ there exists a regular cardinal λ such that $\widehat{\yoneda}$ factors as

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\widehat{\yoneda}} & \mathrm{Ind}(\mathcal{C}^\lambda) \\ & \searrow^{\widehat{\yoneda}} & \downarrow \\ & & \mathrm{Ind}(\mathcal{C}) \end{array}$$

Now if $\mathcal{C}$ is dualizable, then we can choose λ to be ω_1 or κ.

Definition 2.3 ([Efi26, Def. 2.58])**.** Let $\mathcal{C} \in \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$. Then there exists a cofiber sequence

$$\mathcal{C} \xrightarrow{\widehat{\yoneda}} \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C})$$

in $\mathrm{Pr}^L_{\mathrm{st}}$. We define the continuous Calkin category $\mathrm{Calk}^{\mathrm{cont}}_\kappa(\mathcal{C}) \coloneqq (\mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C}))^{\omega_0}$ to be the compact objects of the cofiber.

This construction induces a functor $\mathrm{Calk}^{\mathrm{cont}}_\kappa\colon \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{perf}}$. This uses that the left square is well-defined and commutes for any strongly continuous functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\widehat{\yoneda}} & \mathrm{Ind}(\mathcal{C}^\kappa) & \longrightarrow & \mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C}) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathrm{Ind}(\mathcal{F}^\kappa)} & & \downarrow{\scriptstyle \overline{\mathcal{F}}} \\ \mathcal{D} & \xrightarrow{\widehat{\yoneda}} & \mathrm{Ind}(\mathcal{D}^\kappa) & \longrightarrow & \mathrm{Ind}(\mathcal{D}^\kappa)/\widehat{\yoneda}(\mathcal{D}) \end{array}$$

Therefore it remains to show that $\overline{\mathcal{F}}\colon \mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}^\kappa)/\widehat{\yoneda}(\mathcal{D})$ preserves compact objects or, equivalently, is strongly continuous. Here it suffices to show that $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ is closed under colimits in $\mathrm{Pr}^L_{\mathrm{st}}$, which we prove later in Proposition 3.6.

Proposition 2.4. *For any dualizable category $\mathcal{C}$, the quotient $\mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C})$ is compactly generated.*[1] *In particular, $\mathrm{Ind}(\mathrm{Calk}^{\mathrm{cont}}_\kappa(\mathcal{C}))$ is canonically equivalent to $\mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C})$.*

Proof. Here $\mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C})$ is the fiber of the colimit-preserving functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$, hence the right adjoint $\iota\colon \mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C}) \hookrightarrow \mathrm{Ind}(\mathcal{C}^\kappa)$ of the quotient functor p preserves colimits. So p preserves compact objects and the image of the composite $\mathcal{C}^\kappa \xhookrightarrow{\yoneda} \mathrm{Ind}(\mathcal{C}^\kappa) \xrightarrow{p} \mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C})$ forms a set S of compact objects. Moreover, any object x in $\mathrm{Ind}(\mathcal{C}^\kappa)/\widehat{\yoneda}(\mathcal{C})$ is equivalent to $p(\iota(x))$ where $\iota(x)$ is a filtered colimit of representables. Hence $p(\iota(x))$ is a filtered colimit of objects from S. □

[1] We have essentially seen this already in the second talk.

Next we show that the continuous Calkin category extends the classical Calkin construction.

Proposition 2.5 ([Efi26, Prop. 2.59])**.** *For any compactly generated, stable category $\mathcal{C}$, there exists a natural equivalence* $\mathrm{Calk}_\kappa(\mathcal{C}^{\omega_0}) \simeq \mathrm{Calk}_\kappa^{\mathrm{cont}}(\mathcal{C})$.

Proof. Here we can identify $\widehat{\text{ɟ}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C}^\kappa)$ with $\mathrm{Ind}(\text{ɟ} : \mathcal{C}^{\omega_0} \to \mathcal{C}^\kappa)$ and use that $\mathrm{Ind}(-)$ and $(-)^{\omega_0}$ are inverse equivalences. □

Our next goal is to show that the continuous Calkin category functor preserves short exact sequences.

Definition 2.6. A short exact sequence in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ is a fiber-cofiber sequence in $\mathrm{Pr}_{\mathrm{st}}^L$.

Remark 2.7. Short exact sequences in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ are also fiber-cofiber sequences in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$. We prove in Proposition 3.6, that the cofiber in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ agrees with the cofiber in $\mathrm{Pr}_{\mathrm{st}}^L$. Furthermore, we will see in the next talk, that fibers of localizations in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ are computed in $\mathrm{Pr}_{\mathrm{st}}^L$ as shown in Efimov [Efi26, Prop. 2.86.2].

Proposition 2.8 ([Efi26, Prop. 2.61])**.** *The functor* $\mathrm{Calk}_\kappa^{\mathrm{cont}}(-) : \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathrm{Cat}^{\mathrm{perf}}$ *preserves short exact sequences.*

Proof. Let $\mathcal{C} \overset{\mathcal{F}}{\hookrightarrow} \mathcal{D} \twoheadrightarrow \mathcal{E}$ be a short exact sequence in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$. It is shown in Ramzi [Ram24, Prop. A.26] that $\mathcal{C}^\kappa \hookrightarrow \mathcal{D}^\kappa \twoheadrightarrow \mathcal{E}^\kappa$ is a Verdier sequence or, equivalently, that $\mathrm{Ind}(\mathcal{C}^\kappa) \hookrightarrow \mathrm{Ind}(\mathcal{D}^\kappa) \twoheadrightarrow \mathrm{Ind}(\mathcal{E}^\kappa)$ is a short exact sequence. Hence it follows that $\mathrm{Calk}_\kappa^{\mathrm{cont}}(\mathcal{C}) \to \mathrm{Calk}_\kappa^{\mathrm{cont}}(\mathcal{D}) \twoheadrightarrow \mathrm{Calk}_\kappa^{\mathrm{cont}}(\mathcal{E})$ is a cofiber sequence, as colimits commute with colimits.

So it remains to show that it preserves fully faithful functors. Here we use that $\mathrm{Ind}(-)$ preserves and detects fully faithful functors, i.e. it suffices to show that $\overline{\mathcal{F}} := \mathrm{Ind}(\mathrm{Calk}_\kappa^{\mathrm{cont}}(\mathcal{F}))$ is fully faithful.

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\widehat{\text{ɟ}}} & \mathrm{Ind}(\mathcal{C}^\kappa) & \longrightarrow & \mathrm{Ind}(\mathrm{Calk}_\kappa^{\mathrm{cont}}(\mathcal{C})) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathrm{Ind}(\mathcal{F}^\kappa)} & & \downarrow{\scriptstyle \overline{\mathcal{F}}} \\ \mathcal{D} & \xrightarrow{\widehat{\text{ɟ}}} & \mathrm{Ind}(\mathcal{D}^\kappa) & \longrightarrow & \mathrm{Ind}(\mathrm{Calk}_\kappa^{\mathrm{cont}}(\mathcal{D})) \end{array}$$

Now passing to horizontal right adjoints gives a diagram,

$$\begin{array}{ccc} \mathcal{C} & \xleftarrow{\mathrm{colim}} & \mathrm{Ind}(\mathcal{C}^\kappa) \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathrm{Ind}(\mathcal{F}^\kappa)} \\ \mathcal{D} & \xleftarrow{\mathrm{colim}} & \mathrm{Ind}(\mathcal{D}^\kappa) \end{array}$$

which commutes by the universal property of Ind.

Hence we can identify $\overline{\mathcal{F}}$ as the fiber of the vertical, fully faithful functors, hence $\overline{\mathcal{F}}$ is also fully faithful. □

3. Colimits of Dualizable Categories

Definition 3.1. We define $\mathrm{Pr}_{\mathrm{st}}^{LL}$ to be the non-full, wide subcategory of $\mathrm{Pr}_{\mathrm{st}}^L$ whose morphisms are precisely the strongly continuous functors.

Remark 3.2. We have full subcategory inclusions $\mathrm{Pr}_{\mathrm{st},\omega_0}^L \hookrightarrow \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \hookrightarrow \mathrm{Pr}_{\mathrm{st}}^{LL}$.

Proposition 3.3 ([Efi26, Prop. 2.53])**.** *Let $\mathcal{D}$ be a presentable, stable category and $\mathcal{F}_i : \mathcal{C}_i \to \mathcal{D}$ a family of strongly continuous functors, where the $\mathcal{C}_i$ are dualizable. Suppose $\mathcal{D}$ is generated under colimits by the union of the essential images of $\mathcal{F}_i$. Then $\mathcal{D}$ is also dualizable.*

Proof. We first reduce to the case of a single functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$: Here the essential images of $\mathcal{F}_i$ and $\coprod \mathcal{C}_i \to \mathcal{D}$ generate the same category. Moreover, dualizable categories are closed under coproducts: Suppose that $\mathcal{C}_i$ is dualizable, i.e. a retract of some compactly generated category $\tilde{\mathcal{C}}_i$. Then $\coprod \tilde{\mathcal{C}}_i$ is also compactly generated as a colimit of compactly generated categories, and in particular, $\coprod \mathcal{C}_i$ is a retract of the compactly generated category $\coprod \tilde{\mathcal{C}}_i$.

Now a stable presentable category is dualizable if and only if arbitrary products distribute over filtered colimits. So consider the square

$$\begin{array}{ccc} \mathrm{Ind}(\mathcal{D}) & \xrightarrow{\mathrm{colim}^{\mathcal{D}}} & \mathcal{D} \\ \downarrow{\scriptstyle \mathcal{F}^*} & & \downarrow{\scriptstyle \mathcal{F}^R} \\ \mathrm{Ind}(\mathcal{C}) & \xrightarrow{\mathrm{colim}^{\mathcal{C}}} & \mathcal{C} \end{array}$$

which commutes by the universal property of the Ind construction, since both composites preserve filtered colimits. Now $\mathcal{F}^*$ and $\mathrm{colim}^{\mathcal{C}}$ preserve limits, hence so does the composition. But as $\mathcal{F}^R$ preserves limits and is conservative, this implies that $\mathrm{colim}^{\mathcal{D}}$ preserves limits, i.e. arbitrary products distribute over filtered colimits. □

Definition 3.4 ([Efi26, Def. 2.54])**.** Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a functor in $\mathrm{Pr}^L_{\mathrm{st}}$. Then the image $\mathrm{Im}(\mathcal{F})$ is the localizing subcategory generated by the image of $\mathcal{F}$, or equivalently, the fiber of $\mathcal{D} \to \mathrm{cofib}(\mathcal{F})$ in $\mathrm{Pr}^L_{\mathrm{st}}$.

Corollary 3.5 ([Efi26, Cor. 2.55])**.** *Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a strongly continuous functor, where $\mathcal{C}$ is a dualizable category. Then*

(a) $\mathrm{Im}(\mathcal{F})$ *is also dualizable,*
(b) the functor $\mathrm{Im}(\mathcal{F}) \to \mathcal{D}$ *is strongly continuous.*

Proof.

(a) This follows from Proposition 3.3.
(b) The functor $\mathrm{Im}(\mathcal{F}) \to \mathcal{D}$ is strongly continuous, since the right adjoint $\mathrm{Im}(\mathcal{F}) \to \mathcal{C}$ is conservative and the composition $\mathcal{D} \to \mathrm{Im}(\mathcal{F}) \to \mathcal{C}$ is colimit-preserving, hence $\mathrm{Im}(\mathcal{F}) \to \mathcal{C}$ also preserves colimits. □

Proposition 3.6 ([Efi26, Prop. 1.7, Prop 2.63])**.** *The categories* $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \hookrightarrow \mathrm{Pr}^{LL}_{\mathrm{st}} \rightarrowtail \mathrm{Pr}^L_{\mathrm{st}}$ *are closed under colimits and hence also cocomplete.*

Proof. We first show that $\mathrm{Pr}^{LL}_{\mathrm{st}}$ is closed under colimits in $\mathrm{Pr}^L_{\mathrm{st}}$ and then that $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ is closed under colimits.

For any diagram $I \to \mathrm{Pr}^{LL}_{\mathrm{st}} \rightarrowtail \mathrm{Pr}^L_{\mathrm{st}}$, we can identify the colimit in $\mathrm{Pr}^L_{\mathrm{st}}$ as the limit in $\mathrm{Pr}^R_{\mathrm{st}}$ or equivalently Cat. Now each transition functor in the limit preserves colimits by assumption, hence so does each projection $\lim \mathcal{C}_i \to \mathcal{C}_i$. This is then equivalent to $\mathcal{C}_i \to \mathrm{colim}\, \mathcal{C}_i$ being strongly continuous. Moreover, any cocone consisting of strongly continuous functors induces a strongly continuous functor out of $\lim \mathcal{C}_i$, since colimits in $\lim \mathcal{C}_i$ are computed pointwise.

Now suppose $I \to \mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ restricts to dualizable categories. Then the coprojections $\mathcal{C}_i \to \mathrm{colim}_I \mathcal{C}_i$ generate $\mathrm{colim}_I \mathcal{C}_i$ under colimits since the functors $\mathrm{colim}_I \mathcal{C}_i \simeq \lim_I \mathcal{C}_i \to \mathcal{C}_i$ are jointly conservative. Hence $\mathrm{colim}_I \mathcal{C}_i$ is dualizable by Proposition 3.3. □

This also finishes the proof that the continuous Calkin category construction from Definition 2.3 is functorial. On the other hand, a new subtlety arises. While filtered colimits in $\mathrm{Cat}^{\mathrm{perf}}$ preserve short exact sequences,[2] it is not true that filtered colimits in $\mathrm{Pr}^L_{\mathrm{st}}$ preserve short exact sequences, as shown in Efimov [Efi26, Remark 2.69].

Thus we want and need to show that filtered colimits in $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ – which are computed in $\mathrm{Pr}^L_{\mathrm{st}}$ – preserve short exact sequences.

[2] in fact, taking filtered colimits is an exact functor

Corollary 3.7 ([Efi26, Cor. 2.64])**.** *Epimorphisms in* $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$ *and* $\mathrm{Pr}_{\mathrm{st}}^{LL}$ *are precisely localizations.*

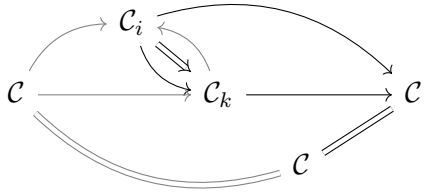
Lemma 3.8 ([Efi26, Prop. 2.65])**.** *Let* $I \to \mathrm{Pr}_{\mathrm{st}}^{L}$ *be a filtered diagram and let* $\mathcal{C} \coloneqq \mathrm{colim}_I^{\mathrm{Pr}_{\mathrm{st}}^{L}} \mathcal{C}_i$ *be the colimit in* $\mathrm{Pr}_{\mathrm{st}}^{L}$*. Let* $\mathcal{F}_i$ *be the coprojections and* $\mathcal{F}_{i,k}\colon \mathcal{C}_i \to \mathcal{C}_k$ *the transition functors. Denote by* $\mathcal{F}_i^R$ *and* $\mathcal{F}_{i,k}^R$ *the corresponding right adjoints.*

Then there exist

(a) an equivalence $\mathrm{colim}_I \mathcal{F}_i\mathcal{F}_i^R \simeq \mathrm{id}$*,*

(b) an equivalence $\mathrm{colim}_{k\geq i,j} \mathcal{F}_{j,k}^R\mathcal{F}_{i,k} \simeq \mathcal{F}_j^R\mathcal{F}_i$ *if each* $\mathcal{F}_{i,j}$ *is strongly continuous.*

Proof. We start with (a): Assume that we have $i \to k$ in I. Then the statement is precisely that the downward-facing natural transformations in



form a colimit cocone. Here the gray arrows denote the right adjoints.

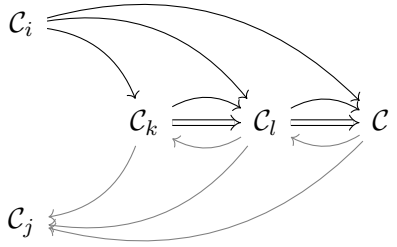
We again use that $\mathcal{C}$ is canonically equivalent to $\lim_I \mathcal{C}_i$ via the projections $\mathcal{F}_i^R\colon \mathcal{C} \to \mathcal{C}_i$. Then we have a chain of equivalences:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(-,-) &\overset{(1)}{\simeq} \lim_I \mathrm{Hom}_{\mathcal{C}_i}(\mathcal{F}_i^R(-), \mathcal{F}_i^R(-)) \\ &\overset{(2)}{\simeq} \lim_I \mathrm{Hom}_{\mathcal{C}}(\mathcal{F}_i\mathcal{F}_i^R(-), -) \\ &\overset{(3)}{\simeq} \mathrm{Hom}_{\mathcal{C}}(\operatorname*{colim}_I \mathcal{F}_i\mathcal{F}_i^R(-), -)\end{aligned}$$

here, (1) follows from the Hom in limit categories, (2) from adjunction and (3) as Hom preserves limits.

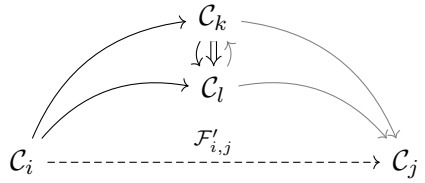
This, together with Yoneda, implies (a).

Now for (b) assume that we have $i, j \to k \to l$ in I. Then the statement from (b) is precisely that the natural transformations to the right in



form a colimit cocone.

Now we have for i, j a functor $\mathcal{F}'_{i,j}\colon \mathcal{C}_i \to \mathcal{C}_j$,



where we take the colimit of the downward-facing natural transformations. In other words, we have $\mathcal{F}'_{i,j}(x) = \mathrm{colim}_{k\geq i,j} \mathcal{F}_{j,k}^R\mathcal{F}_{i,k}(x)$. This induces a well-defined functor

$\mathcal{F}_i' \colon \mathcal{C}_i \to \mathcal{C} \simeq \lim_j \mathcal{C}_j$, since each transition functor $\mathcal{F}^R_{j',j} \colon \mathcal{C}_j \to \mathcal{C}_{j'}$ preserves colimits by assumption.

We show that $\mathcal{F}_i$ and $\mathcal{F}_i'$ are canonically equivalent. This then implies the statement from (b), as

$$\begin{aligned}\mathcal{F}_j^R(\mathcal{F}_i) &\simeq \mathcal{F}_j^R(\operatorname*{colim}_{k\geq i,j'} \mathcal{F}^R_{j',k}\mathcal{F}_{i,k})_{j'} \\ &\simeq \operatorname*{colim}_{k\geq i,j} \mathcal{F}^R_{j,k}\mathcal{F}_{i,k}\end{aligned}$$

Now we finally show $\mathcal{F}_i \simeq \mathcal{F}_i'$ by proving that $\mathcal{F}_i'$ is the left adjoint of $\mathcal{F}_i^R$:

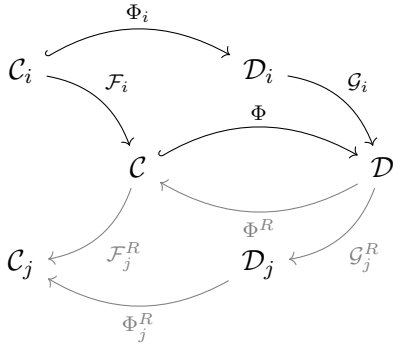
$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}}(\mathcal{F}_i'(-),-) &\overset{(1)}{\simeq} \lim_j \operatorname{Hom}_{\mathcal{C}_j}(\mathcal{F}'_{i,j}(-), \mathcal{F}_j^R(-)) \\ &\overset{(2)}{\simeq} \lim_j \operatorname{Hom}_{\mathcal{C}_j}(\operatorname*{colim}_{k\geq i,j} \mathcal{F}^R_{j,k}\mathcal{F}_{i,k}(-), \mathcal{F}_j^R(-)) \\ &\overset{(3)}{\simeq} \lim_j \lim_{k\geq i,j} \operatorname{Hom}_{\mathcal{C}_j}(\mathcal{F}^R_{j,k}\mathcal{F}_{i,k}(-), \mathcal{F}_j^R(-)) \\ &\overset{(4)}{\simeq} \lim_{k\geq j\geq i} \operatorname{Hom}_{\mathcal{C}_j}(\mathcal{F}^R_{j,k}\mathcal{F}_{i,k}(-), \mathcal{F}_j^R(-)) \\ &\overset{(5)}{\simeq} \lim_{j=k\geq i} \operatorname{Hom}_{\mathcal{C}_j}(\mathcal{F}_{i,j}(-), \mathcal{F}_j^R(-)) \\ &\overset{(6)}{\simeq} \lim_{j\geq i} \operatorname{Hom}_{\mathcal{C}_i}(-, \mathcal{F}^R_{i,j}(\mathcal{F}_j^R(-))) \\ &\overset{(7)}{\simeq} \operatorname{Hom}_{\mathcal{C}_i}(-, \mathcal{F}_i^R(-))\end{aligned}$$

Here (1) follows from Hom in limit categories, (2) writes out the definition of $\mathcal{F}'_{i,j}$, (3) is limit-preserving of Hom, (6) is the adjunction and (7) is commuting of the projections and the transition functors. For (4) and (5) we have used an initial approximation. □

Proposition 3.9 ([Efi26, Prop. 2.67])**.** *We have for* $\Pr^{LL}_{\mathrm{st}}$ *or* $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}}$ *that*

(a) the class of fully faithful functors is closed under filtered colimits,
(b) filtered colimits preserve short exact sequences.

Proof. We show the case of $\Pr^{LL}_{\mathrm{st}}$, the other case follows as $\mathrm{Cat}^{\mathrm{dual}}_{\mathrm{st}} \subseteq \Pr^{LL}_{\mathrm{st}}$ is closed under filtered colimits by Proposition 3.6. Let $(\mathcal{C}_i)$ and $(\mathcal{D}_i)$ be two diagrams in $\Pr^{LL}_{\mathrm{st}}$ indexed by a filtered I and $\Phi_i \colon \mathcal{C}_i \to \mathcal{D}_i$ a natural transformation of pointwise fully faithful and strongly continuous functors. Let $\mathcal{C} \coloneqq \operatorname{colim}_I \mathcal{C}_i, \mathcal{D} \coloneqq \operatorname{colim}_I \mathcal{D}_i$ and $\Phi \coloneqq \operatorname{colim}_I \Phi_i$ be the respective colimits. Then we have a diagram of the form



We want to show that there exists a natural equivalence between id and $\Phi^R\Phi$. For this, we reduce to showing that this natural transformation induces an equivalence between $\mathcal{F}_j^R\mathcal{F}_i$ and $\mathcal{F}_j^R\Phi^R\Phi\mathcal{F}_i$ for all $i, j \in I$: We may postcompose with $\mathcal{F}_j^R$ as they are jointly

conservative. Furthermore, the essential images of $\mathcal{F}_i$ generate $\mathcal{C}$ under colimits. Therefore we may precompose with $\mathcal{F}_i$, since both $\mathcal{F}_j^R$ and $\mathcal{F}_j^R\Phi^R\Phi$ are (compositions) of colimit-preserving functors and hence the collection of objects on which the natural transformation is an equivalence, is closed under colimits.

Here we use

$$\begin{aligned}
\mathcal{F}_j^R\Phi^R\Phi\mathcal{F}_i &\overset{(1)}{\simeq} \Phi_j^R\mathcal{G}_j^R\mathcal{G}_i\Phi_i \\
&\overset{(2)}{\simeq} \Phi_j^R(\operatorname*{colim}_{k\geq i,j}\mathcal{G}_{j,k}^R\mathcal{G}_{i,k})\Phi_i \\
&\overset{(3)}{\simeq} \operatorname*{colim}_{k\geq i,j}\Phi_j^R\mathcal{G}_{j,k}^R\mathcal{G}_{i,k}\Phi_i \\
&\overset{(1)}{\simeq} \operatorname*{colim}_{k\geq i,j}\mathcal{F}_{j,k}^R\Phi_k^R\Phi_k\mathcal{F}_{i,k} \\
&\overset{(4)}{\simeq} \operatorname*{colim}_{k\geq i,j}\mathcal{F}_{j,k}^R\mathcal{F}_{i,k} \\
&\overset{(2)}{\simeq} \mathcal{F}_j^R\mathcal{F}_i
\end{aligned}$$

Here (1) is naturality of Φ, (2) follows from Lemma 3.8 (b), (3) follows as Φ_j^R preserves colimits, or equivalently, Φ_j is strongly continuous and finally (4) is the natural equivalence of the unit from the adjunction $\Phi_k \dashv \Phi_k^R$. □

Proposition 3.10 ([Efi26, Prop. 2.70])**.** *The functor* $(-)^{\omega_0}\colon \mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}} \to \mathrm{Cat}^{\mathrm{perf}}$ *preserves filtered colimits. Equivalently, since* $(-)^{\simeq}\colon \mathrm{Cat}^{\mathrm{perf}} \to \mathrm{An}$ *is conservative,* Sp *is a compact object in* $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$*.*

Proof. Let $(\mathcal{C}_i)$ be a filtered diagram in $\mathrm{Cat}_{\mathrm{st}}^{\mathrm{dual}}$. Then there exists a diagram

$$\begin{array}{ccccc}
\operatorname{colim}_I \mathcal{C}_i & \longrightarrow & \operatorname{colim}_I \mathrm{Ind}(\mathcal{C}_i^{\omega_1}) & \longrightarrow & \operatorname{colim}_I \mathrm{Ind}(\mathrm{Calk}_{\omega_1}^{\mathrm{cont}}(\mathcal{C}_i)) \\
\| & & \downarrow\simeq & & \downarrow\simeq \\
\operatorname{colim}_I \mathcal{C}_i & \longrightarrow & \mathrm{Ind}(\operatorname{colim}_I \mathcal{C}_i^{\omega_1}) & \longrightarrow & \mathrm{Ind}(\operatorname{colim}_I \mathrm{Calk}_{\omega_1}^{\mathrm{cont}}(\mathcal{C}_i))
\end{array}$$

Then we have a chain of equivalences

$$\begin{aligned}
(\operatorname*{colim}_I \mathcal{C}_i)^{\omega_0} &\overset{(1)}{\simeq} \mathrm{fib}(\operatorname*{colim}_I \mathcal{C}_i^{\omega_0} \to \operatorname*{colim}_I \mathrm{Calk}_{\omega_1}^{\mathrm{cont}}(\mathcal{C}_i)) \\
&\overset{(2)}{\simeq} \operatorname*{colim}_I(\mathrm{fib}\,\mathcal{C}_i^{\omega_0} \to \mathrm{Calk}_{\omega_1}(\mathcal{C}_i)) \\
&\overset{(3)}{\simeq} \operatorname*{colim}_I \mathcal{C}_i^{\omega_0}
\end{aligned}$$

where we use for (1) that a fully faithful, strongly continuous inclusion preserves and detects compact objects, for (2) that filtered colimits are exact in $\mathrm{Cat}^{\mathrm{perf}}$ and for (3) Definition 2.1. □

References

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