ON QUILLEN'S PLUS CONSTRUCTION

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Abstract. A discussion of Quillen's plus construction from an ∞-categorical perspective.

Let $\mathcal{X}$ be an ∞-topos. An object $X \in \mathcal{X}$ is called *acyclic* if the map $X \to *$ is an epimorphism in the categorical sense, i.e., if the square

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & * \end{array}$$

is cocartesian. Note that when $\mathcal{X} = \mathcal{S}$, this is equivalent to $\tilde{\mathrm{H}}_*(X, \mathbb{Z}) = 0$. We shall say that a map $X \to Y$ in $\mathcal{X}$ is *acyclic* if it is acyclic as an object of $\mathcal{X}_{/Y}$. The class of acyclic maps is closed under composition, base change, cobase change, colimits, and finite products. Moreover, if $g \circ f$ and f are acyclic, then g is acyclic.

Lemma 1. *Acyclic maps form the left class of a modality on $\mathcal{X}$.*

Proof. It suffices to show that the class of acyclic maps is of small generation as a saturated class. The full subcategory of $\mathcal{X}$ spanned by the acyclic objects is accessible, being the fiber of the suspension functor. It is thus generated under filtered colimits by a small subcategory. Let $\mathcal{C} \subset \mathrm{Fun}(\Delta^1, \mathcal{X})$ be the union of these small subcategories of $\mathcal{X}_{/X}$ as X ranges over a small set of generators of $\mathcal{X}$. Using that acyclic maps are stable under base change, we immediately deduce that $\mathcal{C}$ generates the class of acyclic maps under colimits. □

In particular, every morphism f in $\mathcal{X}$ factors uniquely as $f = h \circ g$ where g is acyclic and h is right orthogonal to acyclic maps. The *plus construction* $X \mapsto X^+$ is the localization functor associated with this factorization system, i.e., $X \to X^+$ is the acyclic map such that X^+ is local with respect to acyclic maps.

For $X \in \mathcal{X}$, recall that $\pi_n(X)$ is a discrete object in $\mathcal{X}_{/X}$, which is a group if $n \geq 1$ (abelian if $n \geq 2$).

Lemma 2 (Hurewicz theorem)**.** *Let $X \in \mathcal{X}$ be an n-connective object for some $n \geq 1$. Then the Hurewicz map $\pi_n(X) \to \mathrm{H}_n(X, \mathbb{Z}) \times X$ in $\mathcal{X}_{/X}$ exhibits $\mathrm{H}_n(X, \mathbb{Z}) \times X$ as the abelianization of $\pi_n(X)$.*

Proof. If $\mathcal{X}$ is a presheaf ∞-topos, this follows from the classical Hurewicz theorem. If $f_* \colon \mathcal{X} \to \mathcal{Y}$ is a geometric morphism and the result holds for some n-connective object $Y \in \mathcal{Y}$, then the result holds for $f^*(Y)$. It remains to observe that X is the preimage by a geometric morphism of an n-connective object in a presheaf ∞-topos $\mathcal{Y}$. Indeed, if $g_* \colon \mathcal{X} \to \mathcal{P}(\mathcal{C})$ is a fully faithful geometric morphism, one can take $\mathcal{Y} = \mathcal{P}(\mathcal{C})_{/\tau_{\leq n-1} g_*(X)} \simeq \mathcal{P}(\mathcal{C}_{/\tau_{\leq n-1} g_*(X)})$. □

Recall that a discrete group is *perfect* if its abelianization is trivial, and *hypoabelian* if it has no nontrivial perfect subgroups.

Lemma 3. *Let $X \in \mathcal{X}$ be acyclic. Then X is 1-connective and $\pi_1(X)$ is perfect. If $\pi_1(X)$ is trivial, then X is ∞-connective.*

Proof. If $\mathcal{C}$ is stable and $F \colon \mathcal{X} \to \mathcal{C}$ preserves pushouts, then clearly $F(X) \simeq F(*)$. In particular, $\tilde{\mathrm{H}}_0(X, \mathbb{Z})$ and $\mathrm{H}_1(X, \mathbb{Z})$ are trivial. The former implies that X is 1-connective. By the latter and the Hurewicz theorem, $\pi_1(X)$ is perfect. The final statement follows immediately from the Blakers–Massey theorem. □

Remark 4. We do not know an example of an ∞-connective acyclic object that is not contractible.

Corollary 5. *Let $f \colon X \to Y$ be an acyclic morphism in $\mathcal{X}$. If $\pi_1(X)$ is hypoabelian, then f is ∞-connective. In particular, if $\mathcal{X}$ is moreover hypercomplete, then $X \simeq X^+$.*

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Proof. It follows from the exact sequence

$$f^*\pi_2(Y) \to \pi_1(f) \to \pi_1(X)$$

that $\pi_1(f)$ is hypoabelian. By Lemma 3, we conclude that $\pi_1(f)$ is trivial, hence that f is ∞-connective. □

Lemma 6 (van Kampen theorem)**.** *Let*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Z & \xrightarrow[f']{} & W \end{array}$$

be a pushout square in $\mathcal{X}$ where f and g induce isomorphisms on $\tau_{\leq 0}$ and let $h = g' \circ f$. Then

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & f^*\pi_1(Y) \\ \downarrow & & \downarrow \\ g^*\pi_1(Z) & \longrightarrow & h^*\pi_1(W) \end{array}$$

is a pushout square of groups.

Proof. Replacing $\mathcal{X}$ by $\mathcal{X}_{/\tau_{\leq 0}X}$, we may assume that X, Y, Z, and hence W are 1-connective. As X is in particular 0-connective, we may assume that it has a global section $s\colon * \to X$. Then $\tau_{\leq 1}X \simeq \mathrm{B}s^*\pi_1(X)$, and similarly for Y, Z, and W. Since B induces an equivalence of categories between discrete groups and pointed 1-connective 1-truncated objects, we deduce that s^* of the given square of groups is a pushout square. But s is 0-connective, so this suffices. □

Lemma 7. *Let $\mathcal{X}$ be a hypercomplete ∞-topos, let $f\colon X \to Y$ be an acyclic morphism in $\mathcal{X}$, and let P be the kernel of $\pi_1(X) \to f^*\pi_1(Y)$. Then f is the initial morphism that kills P.*

Proof. Let $g\colon X \to X'$ be a morphism that kills P, and consider the pushout square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow \\ X' & \xrightarrow[f']{} & Y'. \end{array}$$

Then f' is acyclic, and we must show that it is an equivalence. Write $g = g'' \circ g'$ where g' is 1-connective and g'' is 0-truncated. Then g' still kills P, so we can assume that g is 1-connective. Since f is acyclic, it is 1-connective by Lemma 3. By the van Kampen theorem, the associated square of groups

$$\begin{array}{ccc} \pi_1(X) & \longrightarrow & \pi_1(X)/P \\ \downarrow & & \downarrow \\ g^*\pi_1(X') & \longrightarrow & g^*f'^*\pi_1(Y') \end{array}$$

is a pushout square. It follows that the lower horizontal map is an isomorphism, so that $g^*\pi_1(f')$ is abelian. It is also perfect by Lemma 3, hence trivial. Since g is 0-connective, $\pi_1(f')$ is trivial. By Lemma 3, we deduce that f' is ∞-connective, whence an equivalence. □

Let $\mathcal{X}^\diamondsuit$ be the ∞-category of pairs (X, P) where $X \in \mathcal{X}$ and P is a perfect subgroup of $\pi_1(X)$; a morphism $(X, P) \to (Y, Q)$ is a morphism $f\colon X \to Y$ sending P to Q.

We say that π_1 *preserves products* if, for every family of objects $(X_\alpha)_\alpha$ with product X, the canonical map $\pi_1(X) \to \prod_\alpha p_\alpha^*\pi_1(X_\alpha)$ is an equivalence, where $p_\alpha\colon X \to X_\alpha$ is the projection. This condition holds in any presheaf ∞-topos, and is inherited by essential subtopoi.

Theorem 8. *Let $\mathcal{X}$ be an ∞-topos where π_1 preserves products. Then the fully faithful functor $\mathcal{X} \hookrightarrow \mathcal{X}^\diamondsuit$, $X \mapsto (X, 1)$, admits a left adjoint $(X, P) \mapsto X/P$. Moreover, the unit map $\eta\colon X \to X/P$ is acyclic.*

Proof. Fix $(X,P)\in\mathcal{X}^\diamondsuit$, and let $\mathcal{X}_{(X,P)/}$ be the full subcategory of $\mathcal{X}_{X/}$ consisting of the morphisms $f\colon X\to Y$ that kill P. Clearly, $\mathcal{X}_{(X,P)/}$ is accessible, and in particular it has a small coinitial subcategory. It therefore suffices to show that $\mathcal{X}_{(X,P)/}$ is closed under nonempty limits in $\mathcal{X}_{X/}$. The assumption that π_1 preserves products implies that $\mathcal{X}_{(X,P)/}$ is closed under products. It remains to show that $\mathcal{X}_{(X,P)/}$ is closed under pullbacks. Given a cartesian square

$$\begin{array}{ccc} Y & \longrightarrow & Y_0 \\ \downarrow & & \downarrow \\ Y_1 & \longrightarrow & Y_{01} \end{array}$$

under X such that P is killed in $\pi_1(Y_0)$ and $\pi_1(Y_1)$, we must show that P is killed in $\pi_1(Y)$. The exact sequence

$$\pi_2(Y_{01})\to\pi_1(Y)\to\pi_1(Y_0)\times\pi_1(Y_1)$$

in $\mathcal{X}_{/Y}$ shows that the image of P in $\pi_1(Y)$ is abelian, hence trivial since P is perfect. The fact that η is acyclic follows immediately from its universal property. □

Remark 9. In the context of Theorem 8, there is an epimorphism $\pi_1(X)\to\eta^*\pi_1(X/P)$ whose kernel is perfect and contains P. We do not know if its kernel equals P in general. This is equivalent to the existence of a morphism $f\colon X\to Y$ that kills exactly P. If $\mathcal{X}=\mathcal{S}$, it is trivial to construct such a morphism where Y is a groupoid. In that case, $X\to X/P$ is an acyclic map that kills exactly P, which is one of the standard characterizations of Quillen's plus construction. It follows that $X\to X/P$ kills exactly P whenever the 2-topos $\mathcal{X}_{\leq 1}$ has enough points.

We will say that $X\in\mathcal{X}$ is *hypoabelian* if the group $\pi_1(X)$ is hypoabelian. As always, a morphism $f\colon X\to Y$ in $\mathcal{X}$ is *hypoabelian* if it is so as an object of $\mathcal{X}_{/Y}$. Since $\pi_1(f)$ is an extension of the kernel of $\pi_1(X)\to f^*\pi_1(Y)$ by a quotient of $f^*\pi_2(Y)$, f is hypoabelian if and only if that kernel is hypoabelian.

Corollary 10. *Let $\mathcal{X}$ be a hypercomplete ∞-topos where π_1 preserves products. Then a morphism in $\mathcal{X}$ is right orthogonal to acyclic morphisms if and only if it is hypoabelian. Hence, for any $X\in\mathcal{X}$, we have $X^+\simeq X/P$ where $P\subset\pi_1(X)$ is the maximal perfect subgroup.*

Proof. One implication was already proved in Corollary 5. Suppose $f\colon X\to Y$ is right orthogonal to acyclic maps. Let K be the kernel of $\pi_1(X)\to f^*\pi_1(Y)$ and let $P\subset K$ be a perfect subgroup. Then f factors uniquely as $X\xrightarrow{\eta}X/P\xrightarrow{g}Y$. Since η is acyclic and f is right orthogonal to acyclic maps, η admits a retraction, hence $\pi_1(X)$ is a retract of $\eta^*\pi_1(X/P)$, hence $P=1$. □

For an arbitrary ∞-topos $\mathcal{X}$, the proof of Theorem 8 shows that $\mathcal{X}_{(X,P)/}$ is closed under nonempty finite limits in $\mathcal{X}_{X/}$, so the inclusion $\mathcal{X}\hookrightarrow\mathcal{X}^\diamondsuit$ admits a left pro-adjoint $\mathcal{X}^\diamondsuit\to\mathrm{Pro}(\mathcal{X})$, $(X,P)\mapsto X/P$. Moreover, if $f_*\colon\mathcal{Y}\to\mathcal{X}$ is a geometric morphism of ∞-topoi and $(X,P)\in\mathcal{X}^\diamondsuit$, we have $f^*(X/P)\simeq f^*(X)/f^*(P)$ by comparison of universal properties. For example, suppose $\mathcal{X}$ is a subtopos of a presheaf ∞-topos $\mathcal{P}(\mathcal{C})$, and let $a\colon\mathcal{P}(\mathcal{C})\to\mathcal{X}$ be the left adjoint to the inclusion. If $X\in\mathcal{X}$ and P is a perfect subgroup of the fundamental group of X in $\mathcal{P}(\mathcal{C})$, then $X/a(P)$ exists in $\mathcal{X}$, since $X/a(P)\simeq a(X/P)$.

Example 11. Let $\mathcal{C}$ be some category of qcqs schemes equipped with the Zariski topology. Let $\mathrm{Vect}(X)$ be the groupoid of finite locally free sheaves on X, and let $\mathrm{sVect}(X)$ be the colimit of the sequence

$$\mathrm{Vect}(X)\xrightarrow{\oplus\mathcal{O}_X}\mathrm{Vect}(X)\xrightarrow{\oplus\mathcal{O}_X}\cdots.$$

Then $\mathrm{sVect}\in\mathrm{Shv}(\mathcal{C})$ and $\pi_1(\mathrm{sVect})(X)_\xi$ is the group $\mathrm{GL}(\xi)$ of automorphisms of $\xi\in\mathrm{sVect}(X)$. Let $\mathrm{SL}\subset\mathrm{GL}$ be the subgroup of automorphisms of determinant 1, which is the subsheaf of GL generated by elementary matrices. Then SL is a perfect subgroup of $\pi_1(\mathrm{sVect})$ and

$$\mathrm{sVect}/\mathrm{SL}\simeq\mathrm{K}$$

in $\mathrm{Shv}(\mathcal{C})$, where K is Thomason–Trobaugh K-theory. In other words, if $F\colon\mathcal{C}^{\mathrm{op}}\to\mathcal{S}$ is a Zariski sheaf and $f\colon\mathrm{sVect}\to F$ is a map that kills SL (e.g., $\pi_1(F)$ is abelian or F is $\mathbb{A}^1$-homotopy invariant), then f factors uniquely through K.

We conclude with a version of the McDuff–Segal group completion theorem.

Theorem 12. *Let $\mathcal{X}$ be an ∞-topos, $M = \coprod_{n\geq 0} M_n$ an $\mathbb{N}$-graded homotopy-commutative monoid in $\mathcal{X}$ with $\tau_{\leq 0}(M) \simeq \mathbb{N}$, and $x\colon * \to M_1$ a global section. Let M^{gp} be the group completion of M, let*

$$M_\infty = \operatorname{colim}(M_0 \xrightarrow{x} M_1 \xrightarrow{x} M_2 \to \cdots),$$

and let $P \subset \pi_1(M_\infty)$ be the commutator subgroup. Then P is perfect and the canonical map $\mathbb{Z}\times M_\infty \to M^{\mathrm{gp}}$ induces an equivalence $\mathbb{Z} \times M_\infty/P \simeq M^{\mathrm{gp}}$. In particular, $\mathbb{Z} \times M_\infty \to M^{\mathrm{gp}}$ is acyclic.

Proof. When $\mathcal{X} = \mathcal{S}$, the classical group completion theorem states that $\mathbb{Z} \times M_\infty \to M^{\mathrm{gp}}$ is acyclic. Since $\pi_1(M^{\mathrm{gp}})$ is abelian, this implies that P is perfect and that $\mathbb{Z} \times M_\infty/P \simeq M^{\mathrm{gp}}$ (by Lemma 7). This immediately generalizes to the case of a presheaf ∞-topos. In general, suppose that $\mathcal{X}$ is a subtopos of $\mathcal{P}(\mathcal{C})$, and let $a\colon \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ be the left adjoint to the inclusion. The graded pieces M_n assemble into an $\mathbb{N}$-graded monoid M' in $\mathcal{P}(\mathcal{C})$ such that $a(M') \simeq M$. The section x defines a morphism of $\mathbb{N}$-graded monoids $\mathbb{N} \to M'$, and we let $M'' \subset M'$ be its image. Then $a(M'') \simeq M$ and M'' satisfies the assumptions of the theorem in $\mathcal{P}(\mathcal{C})$. The result for M in $\mathcal{X}$ then follows from the result for M'' in $\mathcal{P}(\mathcal{C})$. $\square$