Homotopy theory and the Bloch–Kato conjecture

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What is homotopy theory?

Homotopy theory is a logical extension of classical mathematics in which equality behaves differently. It is also called higher mathematics.

Equality in classical mathematics

In classical mathematics, the equality of two objects is a truth value.

Fundamental principle of higher mathematics

An equality between two mathematical objects is itself a mathematical object.

This entails in particular that there can be more than one equality between two mathematical objects.

Anima

The most basic concept in homotopy theory is that of *anima*, which is the higher-mathematical analogue of *set*:

Definition

- An anima is any collection of mathematical objects.
- A set is an anima in which equality is a truth value.

By contrast, if X is a general anima, then equalities between two elements $x, y \in X$ form again an anima $\mathrm{Eq}(x, y)$, by the fundamental principle.

Historically, this concept emerged from topology, category theory, and logic. Correspondingly, anima are often called spaces, ∞-groupoids, or types.

However, *anima* is a primitive concept, i.e., it does not reduce to simpler concepts, just like *set* in classical mathematics.

Understanding anima I: Depth

Recursive definition (depth)

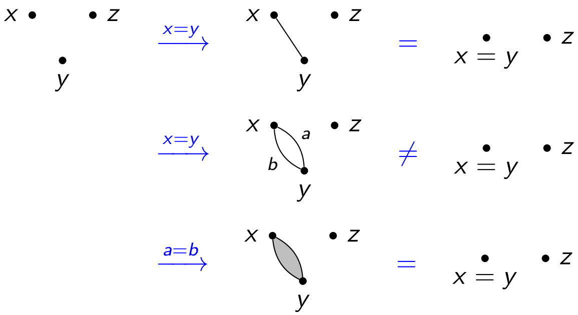
- A one-point set has depth -2.
- An anima X is said to have **depth n** if $\mathrm{Eq}(x, y)$ has depth $n-1$ for all $x, y \in X$.

Equivalently: X has depth $n \Leftrightarrow (n+1)$-equality in X is a truth value.

	depth	collection of objects	categorical structure
classical mathematics	-2	element	"
classical mathematics	-1	truth value	"
classical mathematics	0	set	poset
	1	groupoid	category
	$\vdots$		
	∞	anima	∞-category

Understanding anima II: Dimension

To construct an anima, one can start with a set and add equalities between elements. One may represent this graphically by drawing lines between points:



Recursive definition (dimension)

- An empty set has dimension -1.
- An anima is said to have dimension n if it is obtained from an anima of dimension $n-1$ by adding n-equalities.

Depth vs Dimension

- Sets = anima of depth 0 = anima of dimension 0.
- A 1-dimensional anima has depth 1, but the converse fails.
- Almost all anima of finite depth ≥ 1 have infinite dimension, and almost all anima of finite dimension ≥ 2 have infinite depth.

Example (Spheres)

We define the anima S^n inductively as follows:

- $S^{-1} = \emptyset$
- S^n is the pushout

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & * \\ \downarrow & \text{PO} & \downarrow \\ * & \longrightarrow & S^n. \end{array}$$

Then S^n has dimension n.
It also has depth n for $n \leq 1$, but depth ∞ for $n \geq 2$.

Anima via topology

Homotopy theory began as the subfield of algebraic topology dealing with homotopy invariants of topological spaces. These are functors

$$F\colon \mathrm{Top} \to C$$

sending homotopy equivalences to isomorphisms, e.g., π_*, H_*, etc.

Fact

The ∞-category of anima can be obtained from the category of CW-complexes by forcing the homotopy equivalences to be isomorphisms:

$$\mathrm{Anima} = \mathrm{Top}^{\mathrm{CW}}[\text{htpy equiv}^{-1}].$$

There are many other classical categories C such that $\mathrm{Anima} = C[M^{-1}]$ for a suitable class of morphisms M. We call these models of anima. They allow us to indirectly talk about anima in classical mathematics.

Higher integers: stable homotopy theory

The analogue of $\mathbb{N}$ in higher mathematics is the anima $\mathbb{F}$ of finite sets: it is the free commutative monoid on one element, and happens to have depth 1 (and dimension ∞). The analogue of $\mathbb{Z}$ is then:

Definition

The sphere spectrum $\mathbb{S}$ is the group completion of $\mathbb{F}$. It is a commutative ring, and modules over $\mathbb{S}$ are called spectra.

Theorem (Barratt–Priddy–Quillen–Segal)

$\mathbb{S} = \operatorname{colim}_{n\to\infty} \Omega^n S^n$, where for X a pointed anima, ΩX is the pullback:

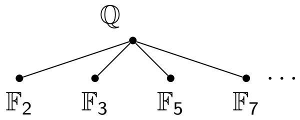
$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & \text{PB} & \downarrow \\ * & \longrightarrow & X. \end{array}$$

Theorem (Pontryagin–Thom)

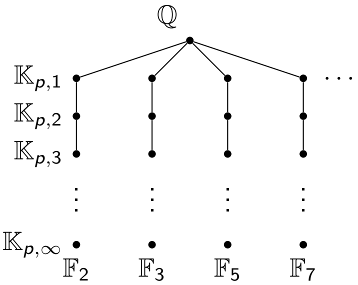
$\mathbb{S} = \mathrm{Cob}^{\mathrm{sfr}}$, the anima of stably framed cobordisms.

Higher primes: chromatic homotopy theory

Spec($\mathbb{Z}$):



Spec($\mathbb{S}$):



Arithmetic fracture square:

$$\begin{array}{ccc} A & \longrightarrow & \prod_p A_p^\wedge \\ \downarrow & \text{PB} & \downarrow \\ A_{\mathbb{Q}} & \longrightarrow & \left(\prod_p A_p^\wedge\right)_{\mathbb{Q}} \end{array}$$

- $\mathbb{K}_{p,n}$ = height n Morava K-theory
- height 1 is complex K-theory

Chromatic convergence:

$A_p^\wedge$ is the limit of the tower

$$\cdots \to L_{p,n+1}A \to L_{p,n}A \to \cdots \to L_{p,1}A$$

Example: the classical Frobenius

Let R be a commutative ring and p a prime number. The p-power map

$$R \xrightarrow{\Delta} R^{\otimes p} \xrightarrow{\text{mult}} R, \quad x \mapsto x^{\otimes p} \mapsto x^p,$$

is not a ring homomorphism, because Δ is not additive.

- Let G be a finite group acting on an abelian group A. The Tate construction $\mathrm{Tate}_G(A)$ is the cokernel of

$$N_G \colon A_G \to A^G, \quad [a] \mapsto \sum_{g \in G} ga.$$

- The Tate diagonal of an abelian group A is the composition

$$\Delta^{\mathrm{Tate}} \colon A \xrightarrow{\Delta} (A^{\otimes p})^{C_p} \to \mathrm{Tate}_{C_p}(A^{\otimes p}).$$

$(x+y)^{\otimes p} - (x^{\otimes p} + y^{\otimes p}) \in \mathrm{im}(N_{C_p}) \ \Rightarrow\ \Delta^{\mathrm{Tate}}$ is additive.

- The Frobenius of R is the ring homomomorphism

$$\mathrm{Frob}_p \colon R \xrightarrow{\Delta^{\mathrm{Tate}}} \mathrm{Tate}_{C_p}(R^{\otimes p}) \xrightarrow{\text{mult}} \mathrm{Tate}_{C_p}(R) = R/p.$$

Example: the higher Frobenius

The construction of $\mathrm{Frob}_p\colon R \to \mathrm{Tate}_{C_p}(R)$ works *mutatis mutandis* when R is a commutative ring spectrum.

Remark (Ambidexterity)

- For vector spaces over $\mathbb{Q}$, the norm N_G is always an isomorphism.
- Over $\mathbb{F}_p = \mathbb{K}_{p,\infty}$, this is only true if p does not divide $|G|$.
- Over $\mathbb{K}_{p,n}$ with $n < \infty$, however, N_G is *always* an isomorphism, even though $p = 0$ in $\mathbb{K}_{p,n}$!

If R is not classical, then $\mathrm{Tate}_{C_p}(R)$ can be p-torsionfree. In this case, the higher Frobenius gives a lift of Frobenius to mixed characteristic (aka an "algebra over $\mathbb{F}_1$").

- If $R = \mathbb{Z}$, then Frob_p encodes the Steenrod operations on ordinary cohomology.
- If $R = \mathbb{K}$, then Frob_p encodes the Adams operations on complex K-theory.
- If $R = \mathbb{S}$, then $\mathrm{Tate}_{C_p}(\mathbb{S})$ is the higher p-adic integers $\mathbb{S}_p^\wedge$.

The Bloch–Kato conjecture

Let k be a field and m an integer coprime to $\mathrm{char}(k)$.
Let k^s be a separable closure of k with Galois group $\mathrm{Gal}(k^s/k)$.

- Milnor K-theory is a quotient of the free associative ring on $k^\times$:

$$K_*^M(k) = \left(\bigoplus_{i=0}^{\infty}(k^\times)^{\otimes i}\right) / \langle a \otimes b = 0 \mid a + b = 1\rangle$$

- Let $\mu_m \subset (k^s)^\times$ be the group of m-th roots of unity.
  The map $k^\times = H^0(\mathrm{Gal}(k^s/k), \mathbb{G}_{\mathrm{mult}}) \to H^1(\mathrm{Gal}(k^s/k), \mu_m)$ induces a ring homomorphism

$$K_*^M(k) \to H^*(\mathrm{Gal}(k^s/k), \mu_m^{\otimes *}).$$

Theorem (Bloch–Kato conjecture 1986, proved by Voevodsky 2010)

$K_*^M(k)/m \to H^*(\mathrm{Gal}(k^s/k), \mu_m^{\otimes *})$ is an isomorphism.

A chromatic proof

Theorem (Bloch–Kato conjecture 1986, proved by Voevodsky 2010)

$K^M_*(k)/m \to H^*(\mathrm{Gal}(k^s/k), \mu_m^{\otimes *})$ is an isomorphism.

- The cases $* = 0, 1$ are trivial.
- The case $* = 2$ is the Merkurjev–Suslin theorem (1982). Their proof used the algebraic K-theory of Quillen as a key ingredient.
- Algebraic K-theory is a higher invariant of schemes:

$$\mathbb{K}^{\mathrm{alg}} \colon \mathrm{Schemes}^{\mathrm{op}} \to \mathrm{Spectra}.$$

  It is an algebraic counterpart of complex K-theory in topology.
- The main insight of Voevodsky was that one could prove Bloch–Kato in degree $n + 1$ if one had an algebraic counterpart of height n Morava K-theory:

$$\mathbb{K}^{\mathrm{alg}}_{p,n} \colon \mathrm{Schemes}^{\mathrm{op}} \to \mathrm{Spectra}.$$

- Voevodsky developed various techniques to construct and study such higher invariants of schemes.
  This is the field of motivic homotopy theory.

Thank you!