Chern character and derived algebraic geometry

Marc Hoyois

January 15, 2009

Hochschild homology

Notations

k is a commutative ring, Alg_k is the category of associative and unital k-algebras, $A \in \mathrm{Alg}_k$

The Hochschild complex $C(A)$ is the simplicial k-module

$$\cdots A \otimes A \otimes A \otimes A \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} A \otimes A \otimes A \substack{\rightarrow\\\rightarrow\\\rightarrow} A \otimes A \rightrightarrows A$$

with

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n & \text{if } 0 \le i \le n-1, \\ a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} & \text{if } i = n, \end{cases}$$

$$s_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = a_0 \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_n.$$

The Hochschild homology of A is $HH_n(A) = \pi_n(C(A))$.

Cyclic modules and mixed complexes

The cyclic category $\Lambda \supset \Delta$

Objects: $[0] = \{0\}$, $[1] = \{0,1\}$, $[2] = \{0,1,2\}$, etc.
Morphisms: compositions of nondecreasing maps and cyclic permutations

A cyclic k-module is a functor $X\colon \Lambda^{\mathrm{op}} \to \mathrm{Mod}_k$.
Equivalently: modules X_n and morphisms $d_i\colon X_n \to X_{n-1}$, $s_i\colon X_n \to X_{n+1}$, $t_n\colon X_n \to X_n$, $n \geq 0$, $0 \leq i \leq n$, satisfying certain relations.
$\rightsquigarrow$ model category $\mathrm{Mod}_k^{\Lambda^{\mathrm{op}}}$ with equivalences defined on the underlying simplicial sets.
A mixed complex (M, b, B) is a nonnegatively graded chain complex (M, b) and a cochain complex (M, B) with $bB + Bb = 0$.
$\rightsquigarrow$ monoidal model category Mix_k with equivalences defined on the underlying chain complexes.

$\exists$ a normalization functor $\mathrm{Mod}_k^{\Lambda^{\mathrm{op}}} \to \mathrm{Mix}_k$ which is a Quillen equivalence

Cyclic homology

The Hochschild complex $C(A)$ has a structure of cyclic k-module with

$$t_n\colon A^{\otimes n+1} \to A^{\otimes n+1}, \quad t_n(a_0 \otimes \cdots \otimes a_n) = a_n \otimes a_0 \otimes \cdots \otimes a_{n-1},$$

whence also a structure of mixed complex by normalization.

Cyclic homology

$HC_*(A) = H_*(k \otimes^{\mathbf{L}} C(A))$ (homotopy orbits)
$HC_*^-(A) = H_*(\mathbf{R}\operatorname{Hom}(k, C(A)))$ (homotopy fixed points)
$\otimes^{\mathbf{L}}$, $\mathbf{R}\operatorname{Hom}$ closed monoidal structure of $\operatorname{Ho}\operatorname{Mix}_k$

There are canonical natural transformations

$$HC^- \to HH \to HC.$$

Morita invariance I

The category Mor_k

Objects: associative and unital k-algebras
Morphisms from A to B: group associated to the monoid of isomorphism classes of left perfect B-A-bimodules
Composition: induced by the tensor product of bimodules

Mor_k is an additive category (the direct sum of A and B is $A \times B$)
$\exists$ functor $\mathrm{Alg}_k \to \mathrm{Mor}_k$:

$$f\colon A \to B \quad \mapsto \quad \text{bimodule } {}_BB_A \text{ with right } A\text{-action from } f.$$

Example

$K_0\colon \mathrm{Alg}_k \to \mathrm{Ab}$ is the composition

$$\mathrm{Alg}_k \to \mathrm{Mor}_k \xrightarrow{\mathrm{Mor}_k(k,?)} \mathrm{Ab}$$

Morita invariance II

The Hochschild complex functor $A \mapsto C(A)$ lifts to Mor_k up to simplicial homotopy: if ${}_BP_A$ is a morphism from A to B in Mor_k, the induced map of cyclic k-modules $C(A) \to C(B)$ is

$$a_0 \otimes \cdots \otimes a_n \mapsto \sum_{j_0,\ldots,j_n} \pi_{j_1}(p^{j_0} a_0) \otimes \pi_{j_2}(p^{j_1} a_1) \otimes \cdots \otimes \pi_{j_0}(p^{j_n} a_n)$$

where $\sum_j \pi_j \otimes p^j \in \mathrm{Hom}_B(P, B) \otimes P$ corresponds to $\mathrm{id}_P \in \mathrm{Hom}_B(P, P)$.

Corollary

The functors HH, HC, and HC^- lift to Mor_k.

The Chern character

Construction of $\mathrm{ch}_0\colon K_0(A) \to HH_0(A) = A/[A,A]$

M a perfect left A-module

- choose an idempotent $e\colon A^n \to A^n$ such that $M \cong \mathrm{Im}(e)$
- take the trace of e, $\mathrm{tr}(e) \in A$
- its image in $A/[A,A]$ depends only on M

Theorem

ch_0 *is Morita natural. There exists a unique Morita natural transformation* ch_0^-, *the* *Chern character*, *making the following diagram commute:*

$$\begin{array}{ccc} K_0 & \xrightarrow{\mathrm{ch}_0} & HH_0 \\ & \underset{\mathrm{ch}_0^-}{\searrow} & \uparrow \\ & & HC_0^- . \end{array}$$

Sheaves on categories

(C, τ) a site (C a category, τ a topology on C)

Presheaves

The functor $h\colon \mathsf{C} \to \mathsf{C}^\wedge$ is initial among functors to cocomplete categories.
$\mathsf{C}^\wedge = \mathsf{Set}^{\mathsf{C}^{\mathrm{op}}}$ is the presheaf category and h is the Yoneda embedding.

Sheaves

The functor $a\colon \mathsf{C}^\wedge \to \mathsf{C}^{\sim,\tau}$ is initial among functors sending τ-local isomorphisms to isomorphisms.

Properties

- a has a fully faithful right adjoint $i\colon \mathsf{C}^{\sim,\tau} \to \mathsf{C}^\wedge$
- The essential image of i is characterized by the usual descent property

Stacks on model categories I

(C, τ) a model site (C a model category, τ a topology on $\operatorname{Ho}\mathsf{C}$)

Prestacks

The functor $h\colon \mathsf{C} \to \mathsf{C}^\wedge$ is homotopy initial among equivalence-preserving functors to model categories.
$\mathbf{R}h$ is the derived Yoneda embedding.

Sheaves

The functor $a\colon \mathsf{C}^\wedge \to \mathsf{C}^{\sim,\tau}$ is initial among left Quillen functors whose total derived functor sends τ-local equivalences to isomorphisms.

Properties

- $(a \rightleftarrows i)$ $\mathbf{R}i\colon \operatorname{Ho}\mathsf{C}^{\sim,\tau} \to \operatorname{Ho}\mathsf{C}^\wedge$ is fully faithful
- The essential image of $\mathbf{R}i$ is characterized by a descent property

Stacks on model categories II

$\mathrm{sSet}^{\mathsf{C}^{\mathrm{op}}}$ is a simplicial model category with equivalences and fibrations defined objectwise.

$\mathsf{C}^{\wedge}$ = left Bousfield localization of $\mathrm{sSet}^{\mathsf{C}^{\mathrm{op}}}$ along the set of morphisms $\{h(x) \to h(y) \mid x \to y$ equivalence in $\mathsf{C}\}$.

Theorem

$\mathrm{Ho}\,\mathsf{C}^{\wedge}$ *is equivalent the the full subcategory of* $\mathrm{Ho}\,\mathrm{sSet}^{\mathsf{C}^{\mathrm{op}}}$ *spanned by equivalence-preserving functors. These are called prestacks.*

A τ-local equivalence is a morphism in $\mathrm{sSet}^{\mathsf{C}^{\mathrm{op}}}$ inducing τ-local isomorphisms on all presheaves of homotopy groups.

$\mathsf{C}^{\sim,\tau}$ = left Bousfield localization of $\mathsf{C}^{\wedge}$ along τ-local equivalences.

Theorem

$\mathrm{Ho}\,\mathsf{C}^{\sim,\tau}$ *is equivalent the the full subcategory of* $\mathrm{Ho}\,\mathrm{sSet}^{\mathsf{C}^{\mathrm{op}}}$ *spanned by equivalence-preserving functors having τ-hyperdescent. These are called stacks.*

Stacks on model categories III

Derived Yoneda lemma

For any $x \in \mathsf{C}$ and $F \in \mathsf{C}^\wedge$ fibrant, there is a natural isomorphism

$$F(x) \cong \mathbf{R}\operatorname{Map}(\mathbf{R}h(x), F)$$

in $\operatorname{Ho}\mathsf{sSet}$. In particular, $\mathbf{R}h$ is fully faithful. Moreover $\mathbf{R}h$ is isomorphic to $x \mapsto \mathbf{R}\operatorname{Map}(x, ?)$.

The topology τ is subcanonical if for all $x \in \mathsf{C}$, $\mathbf{R}h(x)$ has hyperdescent. This is equivalent to the existence of a commutative diagram

$$\begin{array}{ccc} \operatorname{Ho}\mathsf{C} & \xrightarrow{\mathbf{R}h} & \operatorname{Ho}\mathsf{C}^\wedge \\ & \searrow & \uparrow \mathbf{R}\mathrm{id} \\ & & \operatorname{Ho}\mathsf{C}^{\sim,\tau}. \end{array}$$

Derived stacks

Notation

Comm_k category of commutative k-algebras

The étale model site

A family $\{Y_i \to X\}_i$ of morphisms in $\mathrm{Ho}\,\mathrm{sComm}_k^{\mathrm{op}}$ is an étale covering if

- $\pi_*(Y_i) \to \pi_*(X) \times_{\pi_0(X)} \pi_0(Y_i)$ is an isomorphism in $\mathrm{Comm}_k^{\mathrm{op}}$
- $\{\pi_0(Y_i) \to \pi_0(X)\}_i$ is an étale covering in $\mathrm{Comm}_k^{\mathrm{op}}$

This defines a model topology $\mathrm{\acute{e}t}$ on $\mathrm{sComm}_k^{\mathrm{op}}$.

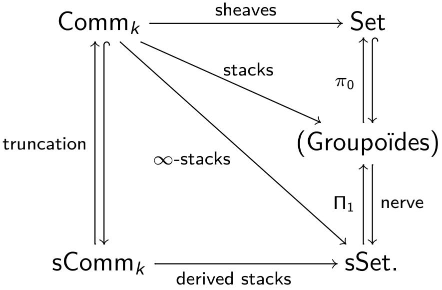
$\mathrm{dSt}_k = (\mathrm{sComm}_k^{\mathrm{op}})^{\sim,\mathrm{\acute{e}t}}$ is the model category of derived stacks

Properties

- $\mathrm{\acute{e}t}$ is subcanonical $\quad\rightsquigarrow\quad \mathbf{R}h\colon\ \mathrm{Ho}\,\mathrm{sComm}_k^{\mathrm{op}} \to \mathrm{Ho}\,\mathrm{dSt}_k$
- $\mathrm{Spec} := \mathbf{R}h$ has a left inverse

$$\mathscr{O}\colon\ \mathrm{Ho}\,\mathrm{dSt}_k \to \mathrm{Ho}\,\mathrm{sComm}_k^{\mathrm{op}},\ F \mapsto \mathbf{R}\,\mathrm{Map}(F, \mathrm{Spec}\,k[T]).$$

The overall picture



Quasi-coherent modules

- $A \in \mathrm{sComm}_k \Rightarrow \mathrm{sMod}_A$ monoidal model category
- $f\colon A \to B$ induces a Quillen adjunction $f_*\colon \mathrm{sMod}_A \rightleftarrows \mathrm{sMod}_B : f^*$
- Quillen invariance: if f is an equivalence, this is a Quillen equivalence

Definition

The derived stack of quasi-coherent modules is

$$\mathrm{Qcoh}\colon \mathrm{sComm}_k \to \mathrm{sSet}, \quad A \mapsto \mathrm{nerve}(\mathrm{sMod}_A^{cw}).$$

Étale descent theorem

Qcoh is a derived stack.

Vector bundles

Vect_A = full subcategory of sMod_A spanned by the dualizable objects in $\mathrm{Ho}\,\mathrm{sMod}_A$

Definition

The derived stack of vector bundles is

$$\mathrm{Vect}\colon \mathrm{sComm}_k \to \mathrm{sSet}, \quad A \mapsto \mathrm{nerve}(\mathrm{Vect}_A^{cw}).$$

Theorem

Vect is a derived stack.

The monoidal $(\infty, 1)$-categories Qcoh_X and Vect_X

$X \in \mathrm{dSt}_k$. There exists a monoidal model category Mod_X of X-modules. These are stacks of modules on the ringed model site of derived stacks over X. The category Mod_X has subcategories

- Qcoh_X of quasi-coherent modules on X and
- Vect_X of vector bundles on X

that can be identified with sMod_A and Vect_A when $X = \operatorname{Spec} A$ is affine.

Theorem

$\mathbf{R}\,\mathrm{Map}(X, \mathrm{Qcoh}) \cong \mathrm{nerve}(\mathrm{Qcoh}_X^w)$ and $\mathbf{R}\,\mathrm{Map}(X, \mathrm{Vect}) \cong \mathrm{nerve}(\mathrm{Vect}_X^w)$.

Proof.

If X is affine, this is the derived Yoneda lemma. An arbitrary X is a homotopy colimits of affine derived stacks, and one proves that the functors $X \mapsto \mathrm{nerve}(\mathrm{Qcoh}_X^w)$ and $X \mapsto \mathrm{nerve}(\mathrm{Vect}_X^w)$ commute with homotopy colimits. □

Geometric interpretation of HH and HC^-

The loop space functor $L\colon \mathrm{dSt}_k \to \mathrm{dSt}_k$ is the right Quillen functor

$$X \mapsto L(X) = X^{B\mathbb{Z}}.$$

Theorem

If A is cofibrant, $L(\operatorname{Spec} A) \simeq \operatorname{Spec} C(A)$.
In particular $HH_n(A) = \pi_n(\mathscr{O}(L(\operatorname{Spec} A)))$

The simplicial group $B\mathbb{Z}$ acts on $L(X)$ by

$$L(X) = X^{B\mathbb{Z}} \to X^{B\mathbb{Z}\times B\mathbb{Z}} \cong (X^{B\mathbb{Z}})^{B\mathbb{Z}} = L(X)^{B\mathbb{Z}}.$$

Hence it also acts on the simplicial algebra $\mathscr{O}(L(X))$. Equivalently, $\mathscr{O}(L(X))$ has a structure of simplicial $k[B\mathbb{Z}]$-module.

Conjecture

If A is cofibrant, $HC_n^-(A) = \pi_n(\mathbf{R}\operatorname{Hom}_{k[B\mathbb{Z}]}(k, \mathscr{O}(L(\operatorname{Spec} A))))$.

Definition of the Chern character I

Observation

$X \in \mathsf{dSt}_k$. A vector bundle on $B\mathbb{Z} \otimes^{\mathbf{L}} X$ gives rise to a vector bundle on X together with an autoequivalence.

Proof.

$[B\mathbb{Z} \otimes^{\mathbf{L}} X, \mathrm{Vect}] \cong [B\mathbb{Z}, \mathbf{R}\,\mathsf{Map}(X, \mathrm{Vect})] \cong [B\mathbb{Z}, \mathsf{nerve}(\mathsf{Vect}_X^w)]$, i.e., an equivalence class of vector bundles on $B\mathbb{Z} \otimes^{\mathbf{L}} X$ is the same thing as a homotopy class of maps $B\mathbb{Z} \to \mathsf{nerve}(\mathsf{Vect}_X^w)$. It induces in particular well-defined maps

$$\pi_0(B\mathbb{Z}) = \{*\} \to [X, \mathrm{Vect}] \quad \rightsquigarrow \quad \text{vector bundle } V \text{ on } X$$
$$\pi_1(B\mathbb{Z}, *) = \mathbb{Z} \to \pi_1(\mathsf{nerve}(\mathsf{Vect}_X^w), V) \cong \mathsf{Aut}_{\mathsf{Ho}\,\mathsf{Vect}_X}(V).$$

□

Definition of the Chern character II

The trace map

$(\mathsf{C}, \otimes, 1)$ rigid monoidal category. The (external) trace map $\mathrm{tr}\colon \mathsf{C}(x, x) \to \mathsf{C}(1, 1)$ is the composition

$$\mathsf{C}(x, x) \cong \mathsf{C}(1, \mathrm{Hom}(x, x)) \cong \mathsf{C}(1, x \otimes x^\vee) \to \mathsf{C}(1, 1).$$

If $X \in \mathsf{dSt}_k$, $\mathrm{Ho}\,\mathsf{Vect}_X$ is a rigid monoidal category.

Construction of the Chern character

V vector bundle on X

- pull back V by the evaluation map $\mathrm{ev}\colon B\mathbb{Z} \otimes^{\mathbf{L}} \mathbf{R}L(X) \to X$;
- $\mathrm{ev}^*(V)$ gives a vector bundle V' on $\mathbf{R}L(X)$ together with an autoequivalence u;
- $\mathrm{tr}(u)$ is the Chern character of V.

Identifying the pair (V', u)

$X \in \mathrm{dSt}_k$ fibrant, V vector bundle on X. $PX = X^{\Delta^1}$ path space of X. There is a canonical map

$$I\colon LX \to PX$$

induced by $\Delta^1 \twoheadrightarrow \Delta^1/\partial\Delta^1 \xrightarrow{\sim} B\mathbb{Z}$. It is a homotopy from the projection

$$p\colon LX \to X$$

to itself.

Observation

$f, g\colon Y \rightrightarrows X$. A homotopy $H\colon f \simeq g$ induces an equivalence $H^*(V)\colon f^*(V) \simeq g^*(V)$.

Proof: Quillen invariance.

Theorem

$V' = p^*(V)$ and $u = I^*(V)$.

The constant case

Theorem

$A \in \mathrm{Comm}_k$, M a perfect A-module. If A is cofibrant in sComm_k, then this construction gives the classical Chern character of M in $HH_0(A)$.

Generalization I

Observation

The construction of the Chern character of a vector bundle on a derived stack X uses only the following properties of the category Vect_X:

- $X \mapsto \mathrm{Vect}_X$ is a presheaf of rigid monoidal $(\infty, 1)$-categories;
- the functor $X \mapsto \mathrm{nerve}(\mathrm{Vect}_X^w)$ is classified by a derived stack $\mathcal{V}\mathrm{ect}$.

Let T be any presheaf defined on sComm_k with values in rigid monoidal $(\infty, 1)$-categories such that

$$T\colon \mathrm{sComm}_k \to \mathrm{sSet}, \quad A \mapsto \mathrm{nerve}(\mathrm{T}(A)^w)$$

is a derived stack. For any derived stack X, it is possible to define a rigid monoidal $(\infty, 1)$-category $\mathrm{T}(X)$ such that

$$\mathbf{R}\,\mathrm{Map}(X, T) \cong \mathrm{nerve}(\mathrm{T}(X)^w).$$

Objects of $\mathrm{T}(X)$ are called *T-objects over X*. Any T-object over X has a Chern character which is an endomorphism of the unit T-object over $\mathbf{R}L(X)$.

Generalization II

The model category sComm_k can be replaced by the category of monoids in any monoidal $(\infty, 1)$-category C:

- $\mathrm{C} = \mathrm{Mod}_k$ gives classical algebraic geometry;
- $\mathrm{C} = \mathrm{Mod}_{\mathscr{O}}$, where $\mathscr{O}$ is a ringed site, gives relative algebraic geometry;
- $\mathrm{C} = \mathrm{sMod}_k$ gives derived algebraic geometry;
- $\mathrm{C} = \mathrm{dgMod}_k$ gives complicial algebraic geometry;
- $\mathrm{C} = \mathrm{Sp}^{\Sigma}$ gives brave new algebraic geometry;
- $\mathrm{C} = \mathrm{Cat}$ gives 2-algebraic geometry;
- etc.

The construction of the Chern character of rigid objects makes sense in any of these geometries.