THE HOMOTOPY FIXED POINTS OF THE CIRCLE ACTION ON HOCHSCHILD HOMOLOGY

MARC HOYOIS

Abstract. We show that Connes' B-operator on a cyclic differential graded k-module M is a model for the canonical circle action on the geometric realization of M. This implies that the negative cyclic homology and the periodic cyclic homology of a differential graded category can be identified with the homotopy fixed points and the Tate fixed points of the circle action on its Hochschild complex.

Let k be a commutative ring and let A be a flat associative k-algebra. The Hochschild complex $\mathrm{HH}(A)$ of A with coefficients in itself is defined as the normalization of a simplicial k-module $A^\natural$ with

$$A^\natural_n = A^{\otimes_k (n+1)}.$$

The simplicial k-module $A^\natural$ is in fact a *cyclic* k-module: it extends to a contravariant functor on Connes' cyclic category Λ. As we will see below, it follows that the chain complex $\mathrm{HH}(A)$ acquires a canonical action of the circle group $\mathbb{T}$. The cyclic homology, negative cyclic homology, and periodic cyclic homology of A over k are classically defined by means of explicit bicomplexes. The goal of this note is to show that:

(1) The *cyclic homology* $\mathrm{HC}(A)$ coincides with the *homotopy orbits* of the $\mathbb{T}$-action on $\mathrm{HH}(A)$.
(2) The *negative cyclic homology* $\mathrm{HN}(A)$ coincides with the *homotopy fixed points* of the $\mathbb{T}$-action on $\mathrm{HH}(A)$.
(3) The *periodic cyclic homology* $\mathrm{HP}(A)$ coincides with the *Tate fixed points* of the $\mathbb{T}$-action on $\mathrm{HH}(A)$.

The first statement is due to Kassel [Kas87, Proposition A.5]. The other two statements are well-known to experts, but their proof seems to be missing from the literature. This gap was mentioned in the introductions to [TV11] and [TV15], and it was partially filled in [TV11], where (2) is proved at the level of connected components for A a smooth commutative k-algebra and $\mathbb{Q} \subset k$.

We will proceed as follows:

- In §1, we recall abstract definitions of cyclic, negative cyclic, and periodic cyclic homology in a more general context, namely for ∞-categories enriched in a symmetric monoidal ∞-category.
- In §2, we show that in the special case of differential graded categories over a commutative ring, the abstract definitions recover the classical ones. The main point is to compare the circle action on Hochschild homology with Connes' B-operator (Theorem 2.3), which in turn boils down to identifying two elements in the second cohomology group of $\mathbb{CP}^\infty$ (Proposition 2.9).

Acknowledgments. I thank Thomas Nikolaus for spotting an error in the proof of Proposition 1.1 in a previous version of this note.

1. Cyclic homology of enriched ∞-categories

Let $\mathcal{E}$ be a presentably *symmetric* monoidal ∞-category, for instance the ∞-category Mod_k for k an E_∞-ring. We denote by $\mathcal{C}\mathrm{at}(\mathcal{E})$ the ∞-category of categorical algebras in $\mathcal{E}$ in the sense of [GH15], i.e., $\mathcal{E}$-enriched ∞-categories with a specified space of objects $\mathrm{ob}(\mathcal{C})$. Let Λ be Connes' cyclic category [Con83], with objects $[n]$ for $n \in \mathbb{N}$. We can associate to every $\mathcal{C} \in \mathcal{C}\mathrm{at}(\mathcal{E})$ a cyclic object $\mathcal{C}^\natural \colon \Lambda^{\mathrm{op}} \to \mathcal{E}$ with

$$\mathcal{C}^\natural_n = \operatorname*{colim}_{a_0,\dots,a_n \in \mathrm{ob}(\mathcal{C})} \mathcal{C}(a_0, a_1) \otimes \cdots \otimes \mathcal{C}(a_{n-1}, a_n) \otimes \mathcal{C}(a_n, a_0).$$

The cyclic category Λ contains the simplex category Δ as a subcategory, and the *Hochschild homology* of $\mathcal{C}$ (with coefficients in itself) is the geometric realization of the restriction of $\mathcal{C}^\natural$ to Δ^{op}:

$$\mathrm{HH}(\mathcal{C}) = \operatorname*{colim}_{n \in \Delta^{\mathrm{op}}} \mathcal{C}^\natural_n \in \mathcal{E}.$$

Date: October 3, 2019.

We refer to [AMGR] for a precise construction of $\mathcal{C}^\natural$ in this context, and for a proof that $\mathrm{HH}(\mathcal{C})$ depends only on the $\mathcal{E}$-enriched ∞-category presented by $\mathcal{C}$ (i.e., the functor HH inverts fully faithful essentially surjective morphisms).

We now recall the canonical circle action on $\mathrm{HH}(\mathcal{C})$. Let $\Lambda \to \tilde{\Lambda}$ denote the ∞-groupoid completion of Λ, and let $\mathbb{T}$ be the automorphism ∞-group of $[0]$ in $\tilde{\Lambda}$. Since Λ is connected, there is a canonical equivalence $B\mathbb{T} \simeq \tilde{\Lambda}$. Let $\mathrm{PSh}(\Lambda, \mathcal{E})$ be the ∞-category of $\mathcal{E}$-valued presheaves on Λ, and let $\mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) \subset \mathrm{PSh}(\Lambda, \mathcal{E})$ be the full subcategory of presheaves sending all morphisms of Λ to equivalences. We have an obvious equivalence

$$\mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) \simeq \mathrm{PSh}(B\mathbb{T}, \mathcal{E}).$$

Since $\mathcal{E}$ is presentable and Λ is small, $\mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E})$ is a reflective subcategory of $\mathrm{PSh}(\Lambda, \mathcal{E})$. We denote by

$$|{-}| \colon \mathrm{PSh}(\Lambda, \mathcal{E}) \to \mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) \simeq \mathrm{PSh}(B\mathbb{T}, \mathcal{E})$$

the left adjoint to the inclusion. The morphisms

$$* \xrightarrow{i} B\mathbb{T} \xrightarrow{p} *$$

each induce three functors between the categories of presheaves. We will write

$$u_{\mathbb{T}} = i^*, \quad (-)_{h\mathbb{T}} = p_!, \quad (-)^{h\mathbb{T}} = p_*$$

for the forgetful functor, the $\mathbb{T}$-orbit functor, and the $\mathbb{T}$-fixed point functor, respectively.

Proposition 1.1. *Let $\mathcal{E}$ be a presentable ∞-category and let $X \in \mathrm{PSh}(\Lambda, \mathcal{E})$ be a cyclic object. There is a natural equivalence*

$$u_{\mathbb{T}}|X| \simeq \operatorname*{colim}_{[n]\in\Delta^{\mathrm{op}}} X([n]).$$

Proof. Let $j\colon \Delta \to \Lambda$ be the inclusion. Let $X \in \mathrm{PSh}_{\simeq}(\Delta, \mathcal{E})$ and let $j_*(X) \in \mathrm{PSh}(\Lambda, \mathcal{E})$ be the right Kan extension of X. We claim that $j_*(X)$ inverts all morphisms in Λ. By the formula for right Kan extension, it suffices to show that following: for every $[n]$, the functor of comma categories $\Delta \times_\Lambda \Lambda_{/[n]} \to \Delta \times_\Lambda \Lambda_{/[0]}$ induced by the unique map $[n] \to [0]$ in Δ is a weak equivalence. Recall that every morphism in Λ can be written uniquely as a composite $h \circ t$ where t is an automorphism and h is in Δ [Lod92, Theorem 6.1.3]. If $\Gamma = \Delta \times_\Lambda \Lambda_{/[0]}$, an object of Γ is a pair $([m], t)$ with $[m] \in \Delta$ and $t \in \mathrm{Aut}_\Lambda([m]) = C_{m+1}$, and a morphism $([m], t) \to ([m'], t')$ is a map $h\colon [m] \to [m']$ in Δ such that $t'ht^{-1}$ is in Δ. It is then clear that the functor

$$\Gamma \to \Delta, \quad ([m], t) \mapsto [m], \quad h \mapsto t'ht^{-1},$$

is a cartesian fibration (whose fibers are sets). Moreover, the functor $\Delta \times_\Lambda \Lambda_{/[n]} \to \Delta \times_\Lambda \Lambda_{/[0]}$ can be identified with the projection $\Gamma \times_\Delta \Delta_{/[n]} \to \Gamma$. Since Δ is cosifted, the forgetful functor $\Delta_{/[n]} \to \Delta$ is coinitial. The pullback of a coinitial functor along a cartesian fibration is still coinitial [Lur17, Proposition 4.1.2.15], so $\Gamma \times_\Delta \Delta_{/[n]} \to \Gamma$ is coinitial and in particular a weak equivalence, as desired.

Thus, we have a commuting square

$$\begin{array}{ccc} \mathrm{PSh}_{\simeq}(\Delta, \mathcal{E}) & \hookrightarrow & \mathrm{PSh}(\Delta, \mathcal{E}) \\ \downarrow j_* & & \downarrow j_* \\ \mathrm{PSh}_{\simeq}(\Lambda, \mathcal{E}) & \hookrightarrow & \mathrm{PSh}(\Lambda, \mathcal{E}). \end{array}$$

Since Δ^{op} is weakly contractible, evaluation at $[0]$ is an equivalence $\mathrm{PSh}_{\simeq}(\Delta, \mathcal{E}) \simeq \mathcal{E}$. The left adjoint square, followed by evaluation at $[0]$, says that $u_{\mathbb{T}}|{-}| \simeq \operatorname{colim} j^*(-)$, as desired. □

It follows from Proposition 1.1 that $u_{\mathbb{T}}|\mathcal{C}^\natural| \simeq \mathrm{HH}(\mathcal{C})$, so that $\mathrm{HH}(\mathcal{C})$ acquires a canonical action of the ∞-group $\mathbb{T}$. As another corollary, we recover the following computation of Connes [Con83, Théorème 10]:

Corollary 1.2. $\tilde{\Lambda} \simeq K(\mathbb{Z}, 2)$.

Proof. If $X \in \mathrm{PSh}_{\simeq}(\Lambda)$, then, by Yoneda, $\mathrm{Map}(\Lambda^0, X) \simeq \mathrm{Map}(\tilde{\Lambda}^0, X)$. In other words, the canonical map $\Lambda^0 \to \tilde{\Lambda}^0$ induces an equivalence $|\Lambda^0| \simeq \tilde{\Lambda}^0$, and hence $u_{\mathbb{T}}|\Lambda^0| \simeq \mathbb{T}$. On the other hand, the underlying simplicial set of Λ^0 is $\Delta^1/\partial\Delta^1$, so $u_{\mathbb{T}}|\Lambda^0| \simeq K(\mathbb{Z}, 1)$ by Proposition 1.1. Thus, $\mathbb{T}$ is a $K(\mathbb{Z}, 1)$, and hence $B\mathbb{T} \simeq \tilde{\Lambda}$ is a $K(\mathbb{Z}, 2)$. □

In particular, $\mathbb{T}$ is equivalent to the circle as an ∞-group, which justifies the notation. If $\mathcal{E}$ is stable, Atiyah duality for the circle provides the *norm map* $\nu_{\mathbb{T}}\colon \Sigma^{\mathfrak{t}}(-)_{h\mathbb{T}} \to (-)^{h\mathbb{T}}$, where $\mathfrak{t}$ is the Lie algebra of $\mathbb{T}$ and $\Sigma^{\mathfrak{t}}$ is suspension by its one-point compactification. Explicitly, if $E \in \mathcal{E}$ has an action of $\mathbb{T}$, the norm is induced by the $(\mathbb{T}\times\mathbb{T})$-equivariant composition

$$\Sigma^{\mathfrak{t}}E \to \Sigma^{\mathfrak{t}}\operatorname{Hom}(\Sigma^\infty_+\mathbb{T}, E) \simeq \Sigma^{\mathfrak{t}}(\Sigma^\infty_+\mathbb{T})^\vee \otimes E \simeq \Sigma^\infty_+\mathbb{T}\otimes E \to E,$$

where the first map is the diagonal, the third is Atiyah duality, and the last is the action. The cofiber of $\nu_{\mathbb{T}}$ is the *Tate fixed point functor* $(-)^{t\mathbb{T}}$.

Definition 1.3. Let $\mathcal{E}$ be a presentably symmetric monoidal ∞-category and let $\mathcal{C} \in \mathcal{C}\mathrm{at}(\mathcal{E})$.

(1) The *cyclic homology* of $\mathcal{C}$ is
$$\mathrm{HC}(\mathcal{C}) = |\mathcal{C}^\natural|_{h\mathbb{T}} \in \mathcal{E}.$$
(2) The *negative cyclic homology* of $\mathcal{C}$ is
$$\mathrm{HN}(\mathcal{C}) = |\mathcal{C}^\natural|^{h\mathbb{T}} \in \mathcal{E}.$$
(3) If $\mathcal{E}$ is stable, the *periodic cyclic homology* of $\mathcal{C}$ is
$$\mathrm{HP}(\mathcal{C}) = |\mathcal{C}^\natural|^{t\mathbb{T}} \in \mathcal{E}.$$

Note that $\mathrm{HC}(\mathcal{C})$ is simply the colimit of $\mathcal{C}^\natural\colon \Lambda^{\mathrm{op}} \to \mathcal{E}$. Note also that $\mathrm{HC}(\mathcal{C})$, $\mathrm{HN}(\mathcal{C})$, and $\mathrm{HP}(\mathcal{C})$ depend only on the $\mathcal{E}$-enriched ∞-category presented by $\mathcal{C}$, since this is the case for $\mathrm{HH}(\mathcal{C})$.

Remark 1.4. There are several interesting refinements of the above definitions. By definition, the invariants $\mathrm{HC}(\mathcal{C})$, $\mathrm{HN}(\mathcal{C})$, and $\mathrm{HP}(\mathcal{C})$ depend only on the circle action on $\mathrm{HH}(\mathcal{C})$. The *topological cyclic homology* of $\mathcal{C}$ is a refinement of negative cyclic homology, defined when $\mathcal{E}$ is the ∞-category of spectra, which uses some finer structure on $\mathrm{HH}(\mathcal{C})$. In another direction, additional structure on $\mathcal{C}$ can lead to $\mathrm{HH}(\mathcal{C})$ being acted on by more complicated ∞-groups. For example, if $\mathcal{C}$ has a duality $\dagger$, then $\mathcal{C}^\natural$ extends to the dihedral category whose classifying space is $BO(2)$. The coinvariants $|\mathcal{C}^\natural|_{h\mathrm{O}(2)}$ are called the *dihedral homology* of $(\mathcal{C},\dagger)$.

The previous definitions apply in particular when $\mathcal{C}$ has a unique object, in which case we may identify it with an A_∞-algebra in $\mathcal{E}$. If A is an E_∞-algebra in $\mathcal{E}$, there is a more direct description of $|A^\natural|$. In this case, $A^\natural$ is the underlying cyclic object of the cyclic E_∞-algebra $\Lambda^0\otimes A \in \mathrm{PSh}(\Lambda, \mathrm{CAlg}(\mathcal{E}))$, where Λ^0 is the cyclic set represented by $[0]\in\Lambda$ and $\otimes$ is the canonical action of the ∞-category $\mathcal{S}$ of spaces on the presentable ∞-category $\mathrm{CAlg}(\mathcal{E})$. For any cyclic space $K \in \mathrm{PSh}(\Lambda)$, we clearly have $|K\otimes A| \simeq |K|\otimes A$. It follows that $|A^\natural| \in \mathrm{PSh}(B\mathbb{T}, \mathcal{E})$ is the underlying object of the E_∞-algebra

$$|\Lambda^0|\otimes A \simeq \mathbb{T}\otimes A \in \mathrm{PSh}(B\mathbb{T}, \mathrm{CAlg}(\mathcal{E})).$$

In particular, $\mathrm{HH}(A)$, $\mathrm{HN}(A)$, and $\mathrm{HP}(A)$ inherit E_∞-algebra structures from A. Their geometric interpretation is the following: if $X = \operatorname{Spec} A$, then $\operatorname{Spec}\mathrm{HH}(A)$ is the *free loop space* of X and $\operatorname{Spec}\mathrm{HN}(A)$ is the *space of circles* in X.

2. Comparison with the classical definitions

Let k be a discrete commutative ring and let A be an A_∞-algebra over k. The cyclic and negative cyclic homology of A over k are classically defined via explicit bicomplexes. Let us start by recalling these definitions, following [Lod92, §5.1].

Let $M_\bullet$ be a cyclic object in an additive category $\mathcal{A}$. The usual presentation of Λ provides the face and degeneracy operators $d_i\colon M_n \to M_{n-1}$ and $s_i\colon M_n\to M_{n+1}$ $(0\le i\le n)$, as well as the cyclic operator $c\colon M_n\to M_n$ of order $n+1$. We define the additional operators

$$\begin{aligned} b\colon M_n \to M_{n-1}, &\quad b = \sum_{i=0}^n (-1)^i d_i,\\ s_{-1}\colon M_n\to M_{n+1}, &\quad s_{-1} = cs_n,\\ t\colon M_n\to M_n, &\quad t = (-1)^n c,\\ N\colon M_n\to M_n, &\quad N = \sum_{i=0}^n t^i,\\ B\colon M_n\to M_{n+1}, &\quad B = (\mathrm{id}-t)s_{-1}N. \end{aligned}$$

We easily verify that $b^2 = 0$, $B^2 = 0$, and $bB + Bb = 0$. In particular, (M, b) is a chain complex in $\mathcal{A}$. We now take $\mathcal{A}$ to be the category Ch_k of chain complexes of k-modules. Then (M, b) is a (commuting) bicomplex and we denote by $(C_*(M), b)$ the total chain complex with

$$C_n(M) = \bigoplus_{p+q=n} M_{p,q}, \quad b = b + (-1)^* d.$$

We then form the (anticommuting) *periodic cyclic bicomplex* $\mathrm{BP}(M)$:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \longleftarrow & & C_2(M) & \xleftarrow{B} & C_1(M) & \xleftarrow{B} & C_0(M) & \longleftarrow \cdots & \\
 & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
\cdots \longleftarrow & & C_1(M) & \xleftarrow{B} & C_0(M) & \xleftarrow{B} & C_{-1}(M) & \longleftarrow \cdots & \\
 & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\
\cdots \longleftarrow & & C_0(M) & \xleftarrow{B} & C_{-1}(M) & \xleftarrow{B} & C_{-2}(M) & \longleftarrow \cdots & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

with $\mathrm{BP}(M)_{p,q} = C_{q-p}(M)$. Removing all the negatively graded columns, we obtain the *cyclic bicomplex* $\mathrm{BC}(M)$; removing all the positively graded columns, we obtain the *negative cyclic bicomplex* $\mathrm{BN}(M)$. Finally, we form the total complexes

$$\mathrm{Tot}\,\mathrm{BC},\ \mathrm{Tot}\,\mathrm{BN},\ \mathrm{Tot}\,\mathrm{BP} \colon \mathrm{PSh}(\Lambda, \mathrm{Ch}_k) \to \mathrm{Ch}_k,$$

where

$$\mathrm{Tot}(B)_n = \operatorname*{colim}_{r\to\infty} \prod_{p\leq r} B_{p,n-p}$$

(i.e., we take the product of the terms towards the upper left corner but their sum towards the lower right corner). These functors clearly preserve quasi-isomorphisms and hence induce functors

$$\mathrm{CC},\ \mathrm{CN},\ \mathrm{CP} \colon \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_k,$$

where Mod_k is the stable ∞-category of k-modules. There is a cofiber sequence

$$\mathrm{CC}[1] \xrightarrow{B} \mathrm{CN} \to \mathrm{CP},$$

where the map "B" is induced by the degree $(0, 1)$ map of bicomplexes $\mathrm{BC}(M) \to \mathrm{BN}(M)$ whose nonzero components are $B \colon C_{i-1}(M) \to C_i(M)$.

Theorem 2.1. *Let k be a discrete commutative ring and $M \in \mathrm{PSh}(\Lambda, \mathrm{Mod}_k)$ a cyclic k-module. Then there are natural equivalences*

$$|M|_{h\mathbb{T}} \simeq \mathrm{CC}(M), \quad |M|^{h\mathbb{T}} \simeq \mathrm{CN}(M), \textit{ and } \quad |M|^{t\mathbb{T}} \simeq \mathrm{CP}(M).$$

In particular, if $\mathcal{C}$ is a k-linear ∞-category, then

$$\mathrm{HC}(\mathcal{C}) \simeq \mathrm{CC}(\mathcal{C}^\natural), \quad \mathrm{HN}(\mathcal{C}) \simeq \mathrm{CN}(\mathcal{C}^\natural), \textit{ and } \quad \mathrm{HP}(\mathcal{C}) \simeq \mathrm{CP}(\mathcal{C}^\natural).$$

We first rephrase the classical definitions in terms of *mixed complexes*, following Kassel [Kas87]. We let $k[\epsilon]$ be the differential graded k-algebra

$$\cdots \to 0 \to k\epsilon \xrightarrow{0} k \to 0 \to \cdots,$$

which is nonzero in degrees 1 and 0. The ∞-category $\mathrm{Mod}_{k[\epsilon]}$ is the localization of the category of differential graded $k[\epsilon]$-modules, also called mixed complexes, at the quasi-isomorphisms. The functors

$$k \otimes_{k[\epsilon]} (-),\ \mathrm{Hom}_{k[\epsilon]}(k, -) \colon \mathrm{Mod}_{k[\epsilon]} \to \mathrm{Mod}_k$$

are related by a *norm map*

$$\nu_\epsilon \colon k[1] \otimes_{k[\epsilon]} (-) \to \mathrm{Hom}_{k[\epsilon]}(k, -),$$

induced by the $k[\epsilon]$-linear map $\epsilon\colon k[1] \to k[\epsilon]$.

We denote by

$$K\colon \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_{k[\epsilon]}$$

the functor induced by sending a cyclic chain complex M to the mixed complex $(C_*(M), b, B)$.

Lemma 2.2. *Let $M \in \mathrm{PSh}(\Lambda, \mathrm{Mod}_k)$. Then*

$$\mathrm{CC}(M) \simeq k \otimes_{k[\epsilon]} K(M) \quad \textit{and} \quad \mathrm{CN}(M) \simeq \mathrm{Hom}_{k[\epsilon]}(k, K(M)),$$

and the norm map $\nu_\epsilon\colon k[1] \otimes_{k[\epsilon]} K(M) \to \mathrm{Hom}_{k[\epsilon]}(k, K(M))$ is identified with $B\colon \mathrm{CC}(M)[1] \to \mathrm{CN}(M)$.

Proof. We work at the level of complexes. Let Qk be the nonnegatively graded mixed complex

$$\cdots \leftrightarrows k\epsilon \underset{0}{\overset{\epsilon}{\leftrightarrows}} k \underset{\epsilon}{\overset{0}{\leftrightarrows}} k\epsilon \underset{0}{\overset{\epsilon}{\leftrightarrows}} k.$$

There is an obvious morphism $Qk \to k$ which is a cofibrant resolution of k for the projective model structure on mixed complexes. By inspection, we have isomorphisms of chain complexes

$$\mathrm{Tot}\,\mathrm{BC}(M) \simeq Qk \otimes_{k[\epsilon]} (C_*(M), b, B) \quad \text{and} \quad \mathrm{Tot}\,\mathrm{BN}(M) \simeq \mathrm{Hom}_{k[\epsilon]}(Qk, (C_*(M), b, B)).$$

This proves the first claim. For C a mixed complex, the norm map ν_ϵ is modeled by the composition

$$Qk[1] \otimes_{k[\epsilon]} C \twoheadrightarrow C/\mathrm{im}(\epsilon)[1] \xrightarrow{\epsilon} \ker(\epsilon|C) \hookrightarrow \mathrm{Hom}_{k[\epsilon]}(Qk, C).$$

The last claim is then obvious. □

Let $k[\mathbb{T}]$ be the A_∞-ring $k \otimes \Sigma^\infty_+\mathbb{T}$. There is an obvious equivalence of ∞-categories

$$\mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) \simeq \mathrm{Mod}_{k[\mathbb{T}]}$$

that makes the following squares commute:

$$\begin{array}{ccc} \mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) & \xrightarrow{(-)_{h\mathbb{T}}} & \mathrm{Mod}_k \\ \simeq\downarrow & & \| \\ \mathrm{Mod}_{k[\mathbb{T}]} & \xrightarrow{k \otimes_{k[\mathbb{T}]} -} & \mathrm{Mod}_k, \end{array} \qquad \begin{array}{ccc} \mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) & \xrightarrow{(-)^{h\mathbb{T}}} & \mathrm{Mod}_k \\ \simeq\downarrow & & \| \\ \mathrm{Mod}_{k[\mathbb{T}]} & \xrightarrow{\mathrm{Hom}_{k[\mathbb{T}]}(k, -)} & \mathrm{Mod}_k. \end{array}$$

Since $\mathbb{T}$ is equivalent to the circle, $H_1(\mathbb{T}, \mathbb{Z})$ is an infinite cyclic group. Let $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ be a generator. Sending ϵ to γ defines an equivalence of augmented A_∞-k-algebras $\gamma\colon k[\epsilon] \simeq k[\mathbb{T}]$, whence an equivalence of ∞-categories

$$\gamma^*\colon \mathrm{Mod}_{k[\mathbb{T}]} \simeq \mathrm{Mod}_{k[\epsilon]}.$$

Moreover, γ^* identifies the norm maps $\nu_\mathbb{T}\colon \Sigma^{\mathfrak{t}}(-)_{h\mathbb{T}} \to (-)^{h\mathbb{T}}$ and $\nu_\epsilon\colon k[1] \otimes_{k[\epsilon]} (-) \to \mathrm{Hom}_{k[\epsilon]}(k, -)$, provided that $\Sigma^{\mathfrak{t}}$ is identified with Σ using the orientation of $\mathbb{T}$ given by γ (since then the Poincaré duality isomorphism $k \simeq H_1(\mathbb{T}, k)$ sends 1 to γ).

The main result of this note is that $K(M)$ is a model for $|M|$. More precisely:

Theorem 2.3. *There exists a generator $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ such that the following triangle commutes:*

$$\begin{array}{ccc} \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) & \xrightarrow{|-|} & \mathrm{PSh}(B\mathbb{T}, \mathrm{Mod}_k) \\ & \searrow^{K} & \simeq \downarrow \gamma^* \\ & & \mathrm{Mod}_{k[\epsilon]}. \end{array}$$

Theorem 2.1 follows from Theorem 2.3 and Lemma 2.2. As an immediate corollary, we recover the following result of Dwyer and Kan [DK85, Remark 6.7]:

Corollary 2.4. *The functor $K\colon \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_{k[\epsilon]}$ induces an equivalence of ∞-categories*

$$\mathrm{PSh}_\simeq(\Lambda, \mathrm{Mod}_k) \simeq \mathrm{Mod}_{k[\epsilon]}.$$

To prove Theorem 2.3, we consider the "universal case", namely the cocyclic cyclic k-module $k[\Lambda^\bullet]$. We have a natural equivalence

$$M \simeq k[\Lambda^\bullet] \otimes_\Lambda M,$$

where

$$\otimes_\Lambda \colon \operatorname{Fun}(\Lambda, \mathrm{Mod}_k) \times \mathrm{PSh}(\Lambda, \mathrm{Mod}_k) \to \mathrm{Mod}_k$$

is the coend pairing. Similarly, we have

$$|M| \simeq |k[\Lambda^\bullet]| \otimes_\Lambda M \quad \text{and} \quad K(M) \simeq K(k[\Lambda^\bullet]) \otimes_\Lambda M,$$

since both $|-|$ and K commute with tensoring with constant k-modules and with colimits (for K, note that colimits in $\mathrm{Mod}_{k[\epsilon]}$ are detected by the forgetful functor to Mod_k). Thus, it will suffice to produce an equivalence of cocyclic $k[\epsilon]$-modules

$$\gamma^*|k[\Lambda^\bullet]| \simeq K(k[\Lambda^\bullet]). \tag{2.5}$$

Let $k[u]$ denote the A_∞-k-coalgebra $k \otimes_{k[\epsilon]} k$. Note that a $k[u]$-comodule structure on $M \in \mathrm{Mod}_k$ is the same thing as map $M \to M[2]$. The functor $k \otimes_{k[\epsilon]} - \colon \mathrm{Mod}_{k[\epsilon]} \to \mathrm{Mod}_k$ factors through a fully faithful functor from $k[\epsilon]$-modules to $k[u]$-comodules:

$$\begin{array}{ccc} & & \mathrm{Comod}_{k[u]} \\ & \nearrow & \downarrow \scriptstyle\text{forget} \\ \mathrm{Mod}_{k[\epsilon]} & \xrightarrow[k \otimes_{k[\epsilon]} -]{} & \mathrm{Mod}_k. \end{array}$$

To prove (2.5), it will therefore suffice to produce an equivalence of cocyclic $k[u]$-comodules

$$k \otimes_{k[\epsilon]} \gamma^*|k[\Lambda^\bullet]| \simeq k \otimes_{k[\epsilon]} K(k[\Lambda^\bullet]). \tag{2.6}$$

Note that both cocyclic objects send all morphisms in Λ to equivalences and hence can be viewed as functors $B\mathbb{T} \to \mathrm{Comod}_{k[u]}$.

Let us first compute the left-hand side of (2.6). The generator γ induces an equivalence of coaugmented A_∞-k-coalgebras $\check\gamma \colon k[u] \simeq k[B\mathbb{T}]$, whence an equivalence of ∞-categories

$$\check\gamma^* \colon \mathrm{Comod}_{k[B\mathbb{T}]} \simeq \mathrm{Comod}_{k[u]}.$$

We clearly have

$$k \otimes_{k[\epsilon]} \gamma^*|k[\Lambda^\bullet]| \simeq \check\gamma^*|k[\Lambda^\bullet]|_{h\mathbb{T}}.$$

Now, $|k[\Lambda^\bullet]|_{h\mathbb{T}} \simeq k[|\Lambda^\bullet|_{h\mathbb{T}}]$, where $|\Lambda^\bullet|_{h\mathbb{T}}$ is a $B\mathbb{T}$-comodule in $\operatorname{Fun}_\simeq(\Lambda, \mathcal{S}) \simeq \mathcal{S}_{/B\mathbb{T}}$. If $\pi^* \colon \mathcal{S} \to \mathcal{S}_{/B\mathbb{T}}$ is the functor $\pi^*X = X \times B\mathbb{T}$, then a $B\mathbb{T}$-comodule structure on π^*X is simply a map $\pi^*X \to \pi^*B\mathbb{T}$, i.e., a map $X \times B\mathbb{T} \to B\mathbb{T}$ in $\mathcal{S}$. Here, $|\Lambda^\bullet|_{h\mathbb{T}}$ is $\pi^*(*) \in \mathcal{S}_{/B\mathbb{T}}$ and its $B\mathbb{T}$-comodule structure $\sigma \colon \pi^*(*) \to \pi^*(B\mathbb{T})$ is given by the identity $B\mathbb{T} \to B\mathbb{T}$. Applying $\check\gamma^*k[-]$, we deduce that the left-hand side of (2.6) is the constant cocyclic k-module $\underline{k}$ with $k[u]$-comodule structure given by the composition

$$\underline{k} \xrightarrow{\sigma} \underline{k}[B\mathbb{T}] \overset{\check\gamma}{\simeq} \underline{k}[u]. \tag{2.7}$$

Note that equivalence classes of $k[u]$-comodule structures on $\underline{k}$ are in bijection with

$$[\underline{k}, \underline{k}[2]] \simeq H^2(B\mathbb{T}, k).$$

Under this classification, (2.7) comes from an integral cohomology class, namely the image of the identity $B\mathbb{T} \to B\mathbb{T}$ under the isomorphism

$$[B\mathbb{T}, B\mathbb{T}] \overset{\check\gamma}{\simeq} H^2(B\mathbb{T}, \mathbb{Z}).$$

In particular, it comes from a generator of the infinite cyclic group $H^2(B\mathbb{T}, \mathbb{Z})$, determined by γ. We must therefore show that the right-hand side of (2.6) is also equivalent to the constant cocyclic k-module $\underline{k}$ with $k[u]$-comodule structure classified by a generator of $H^2(B\mathbb{T}, \mathbb{Z})$.

Recall that $K(k[\Lambda^\bullet])$ is the following mixed complex of cocyclic k-modules:

$$\cdots \rightleftarrows k[\Lambda_2] \underset{B}{\overset{b}{\rightleftarrows}} k[\Lambda_1] \underset{B}{\overset{b}{\rightleftarrows}} k[\Lambda_0].$$

Consider the mixed complex Qk from the proof of Lemma 2.2, which can be used to compute $k \otimes_{k[\epsilon]} -$ at the level of complexes. It comes with an obvious self-map $Qk \to Qk[2]$ which induces the $k[u]$-comodule structure on $k \otimes_{k[\epsilon]} M$ for every mixed complex M. Let us write down explicitly the resulting chain complex $Qk \otimes_{k[\epsilon]} K(k[\Lambda^\bullet])$ of cocyclic $k[u]$-comodules. It is the total complex of the first-quadrant bicomplex

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ k[\Lambda_2] & \xleftarrow{B} & k[\Lambda_1] & \xleftarrow{B} & k[\Lambda_0] \\ {\scriptstyle b}\downarrow & & {\scriptstyle b}\downarrow & & \\ k[\Lambda_1] & \xleftarrow{B} & k[\Lambda_0] & & \\ {\scriptstyle b}\downarrow & & & & \\ k[\Lambda_0], & & & & \end{array} \tag{2.8}$$

with $k[u]$-comodule structure induced by the obvious degree $(-1,-1)$ endomorphism δ.

Proposition 2.9. *The bicomplex* (2.8) *is a resolution of the constant cocyclic k-module $\underline{k}$. Moreover, the endomorphism δ represents a generator of the invertible k-module $[\underline{k}, \underline{k}[2]] \simeq H^2(B\mathbb{T}, k)$.*

Proof. Let K_{**} be the bicomplex (2.8), with the obvious augmentation $K_{**} \to \underline{k}$. For M a cyclic object in an additive category, we define the operator $b' \colon M_n \to M_{n-1}$ by

$$b' = b - (-1)^n d_n = \sum_{i=0}^{n-1} (-1)^i d_i.$$

Let L_{**} be the $(2,0)$-periodic first-quadrant bicomplex

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \\ k[\Lambda_2] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_2] & \xleftarrow{N} & k[\Lambda_2] & \longleftarrow & \cdots \\ {\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & {\scriptstyle b}\downarrow & & \\ k[\Lambda_1] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_1] & \xleftarrow{N} & k[\Lambda_1] & \longleftarrow & \cdots \\ {\scriptstyle b}\downarrow & & {\scriptstyle -b'}\downarrow & & {\scriptstyle b}\downarrow & & \\ k[\Lambda_0] & \xleftarrow{\mathrm{id}-t} & k[\Lambda_0] & \xleftarrow{N} & k[\Lambda_0] & \longleftarrow & \cdots \end{array}$$

with the obvious augmentation $L_{**} \to \underline{k}$, and let M_{**} be the bicomplex obtained from L_{**} by annihilating the even-numbered columns. Let $\phi \colon \operatorname{Tot} K_{**} \to \operatorname{Tot} L_{**}$ be the map induced by $(\mathrm{id}, s_{-1}N) \colon k[\Lambda_n] \to k[\Lambda_n] \oplus k[\Lambda_{n+1}]$, and let $\psi \colon \operatorname{Tot} L_{**} \to \operatorname{Tot} M_{**}$ be the map induced by $-s_{-1}N + \mathrm{id} \colon k[\Lambda_n] \oplus k[\Lambda_{n+1}] \to k[\Lambda_{n+1}]$. A straightforward computation shows that ϕ and ψ are chain maps and that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Tot} K_{**} & \xrightarrow{\phi} & \operatorname{Tot} L_{**} & \xrightarrow{\psi} & \operatorname{Tot} M_{**} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \underline{k} & \xrightarrow[\mathrm{id}]{} & \underline{k} & \longrightarrow & 0 & \longrightarrow & 0. \end{array} \tag{2.10}$$

From the identity $s_{-1}b' + b's_{-1} = \mathrm{id}$, we deduce that each column of M_{**} has zero homology, and hence that that $\operatorname{Tot} M_{**} \simeq 0$. Next we show that each row of L_{**} has zero positive homology, so that the homology of $\operatorname{Tot} L_{**}$ can be computed as the homology of the zeroth column of horizontal homology of L_{**}. This can be proved pointwise, so consider a part of the nth row evaluated at $[m]$:

$$\cdots \to k[\Lambda(n,m)] \xrightarrow{\mathrm{id}-t} k[\Lambda(n,m)] \xrightarrow{N} k[\Lambda(n,m)] \to \cdots . \tag{2.11}$$

By the structure theorem for Λ, we have $\Lambda(n,m) = C_{n+1} \times \Delta(n,m)$, where C_{n+1} is the set of automorphisms of $[n]$ in Λ. Thus, (2.11) is obtained from the complex

$$\cdots \to k[C_{n+1}] \xrightarrow{\mathrm{id}-t} k[C_{n+1}] \xrightarrow{N} k[C_{n+1}] \to \cdots . \tag{2.12}$$

by tensoring with the free k-module $k[\Delta(n,m)]$, and we need only prove that (2.12) is exact. Let

$$x = \sum_{i=0}^{n} x_i c^i \in k[C_{n+1}].$$

Suppose first that $x(\mathrm{id} - t) = 0$; then $x_i = (-1)^{ni} x_0$ and hence $x = x_0 N$. Suppose next that $xN = 0$, i.e., that $\sum_{i=0}^{n} (-1)^{ni} x_{n-i} = 0$; putting $y_0 = x_0$ and $y_i = x_i + (-1)^n y_{i-1}$ for $i > 0$, we find $x = y(\mathrm{id} - t)$. This proves the exactness of (2.12), and also that the image of $\mathrm{id} - t\colon k[C_{n+1}] \to k[C_{n+1}]$ is exactly the kernel of the surjective map $k[C_{n+1}] \to k$, $x \mapsto \sum_{i=0}^{n} (-1)^{ni} x_{n-i}$. This map identifies the 0th homology of the nth row of L_{**} evaluated at $[m]$ with $k[\Delta(n,m)]$. Moreover, the vertical map $k[\Delta(n,m)] \to k[\Delta(n-1,m)]$ induced by $-b$ is the usual differential associated with the simplicial k-module $k[\Delta^m]$. This proves that $\operatorname{Tot} L_{**} \to \underline{k}$ is a resolution of k. From (2.10) we deduce that $\operatorname{Tot} K_{**} \to \underline{k}$ is a quasi-isomorphism.

To prove the second statement, we contemplate the complex $\operatorname{Hom}(\operatorname{Tot} K_{**}, \underline{k})$: it is the total complex of the bicomplex

$$\begin{array}{ccccc}
 & & & & k \\
 & & & & {\scriptstyle 0}\downarrow \\
 & & k & \longleftarrow & k \\
 & & {\scriptstyle 0}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\
k & \longleftarrow & k & \longleftarrow & k \\
{\scriptstyle 0}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle 0}\downarrow \\
\vdots & & \vdots & & \vdots
\end{array}$$

with trivial horizontal differentials and alternating vertical differentials. We immediately check that

$$\operatorname{Tot} K_{**} \xrightarrow{\delta} (\operatorname{Tot} K_{**})[2] \to \underline{k}[2]$$

is a cocycle generating the second cohomology module. □

It follows from Proposition 2.9 that the right-hand side of (2.6) is the constant cocyclic k-module $\underline{k}$ with $k[u]$-comodule structure classified by $\delta\colon \underline{k} \to \underline{k}[2]$. Comparing with (2.7) and noting that δ is natural in k, we deduce that Theorem 2.3 holds by choosing $\gamma \in H_1(\mathbb{T}, \mathbb{Z})$ to be the generator corresponding to $\delta \in H^2(B\mathbb{T}, \mathbb{Z})$.

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