K-THEORY OF DUALIZABLE CATEGORIES (AFTER A. EFIMOV)

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We explain the definition of the K-theory of "large" stable ∞-categories, due to Alexander Efimov. These notes are based on a talk given by Efimov at the CATS5 conference in Lisbon in October 2018.

We start by fixing some terminology. We shall refer to ∞-categories as categories. We write $\mathcal{P}\mathrm{r}_{\mathrm{St}}$ for the category of stable presentable categories and colimit-preserving functors, $\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}$ for the subcategory of dualizable objects and right-adjointable morphisms (with respect to the symmetric monoidal and 2-categorical stuctures of $\mathcal{P}\mathrm{r}_{\mathrm{St}}$), and $\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{cg}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}$ for the subcategory of compactly generated categories and compact functors (i.e., functors whose right adjoints preserve filtered colimits). A *localization sequence* in any such category will mean a cofiber sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ where $\mathcal{A} \to \mathcal{B}$ is fully faithful.

It is well known that $\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{cg}}$ is a full subcategory of $\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}}$. In fact:

Theorem 1 (Lurie)**.** *For $\mathcal{C}$ a stable presentable category, the following are equivalent:*

(1) *$\mathcal{C}$ is dualizable in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$.*
(2) *$\mathcal{C}$ is a retract in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$ of a compactly generated category.*
(3) *The colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint $\hat{y}\colon \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$.*

Note that the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$ is left adjoint to the Yoneda embedding $y\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$, which is fully faithful. Hence, the functor $\hat{y}$ is also fully faithful. When $\mathcal{C}$ is compactly generated, $\hat{y}$ is the Ind-extension of the inclusion $\mathcal{C}^\omega \subset \mathcal{C}$.

Definition 2. Let $\mathcal{C}$ be a presentable stable category. The *Calkin category* $\mathrm{Calk}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C})$ is the kernel of the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}) \to \mathcal{C}$.

Proposition 3. *Suppose $\mathcal{C} \in \mathcal{P}\mathrm{r}_{\mathrm{St}}$ is dualizable. Then:*

(1) $\mathrm{Calk}(\mathcal{C}) = \mathrm{Ind}(\mathrm{Calk}(\mathcal{C})^\omega)$.
(2) *The inclusion $\mathrm{Calk}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C})$ admits a left adjoint*
$$\Phi\colon \mathrm{Ind}(\mathcal{C}) \to \mathrm{Calk}(\mathcal{C})$$
that preserves compact objects.
(3) *There is a localization sequence of cocomplete stable categories*
$$\mathcal{C} \underset{\mathrm{colim}}{\overset{\hat{y}}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}) \overset{\Phi}{\rightleftarrows} \mathrm{Calk}(\mathcal{C}).$$

Proof. The left adjoint Φ is the cofiber of the counit transformation $\hat{y} \circ \mathrm{colim} \to \mathrm{id}$. It preserves compact objects because its right adjoint preserves colimits. Since Φ is essentially surjective, $\mathrm{Calk}(\mathcal{C})$ is generated under colimits by $\Phi(\mathcal{C}) \subset \mathrm{Calk}(\mathcal{C})^\omega$, and the first assertion follows. □

Note that $\mathrm{Calk}(\mathcal{C})^\omega$ is a locally small but *large* category, so that $\mathrm{Calk}(\mathcal{C})$ is not presentable (unless $\mathcal{C} = 0$).

Definition 4. Let $\mathcal{C} \in \mathcal{P}\mathrm{r}_{\mathrm{St}}$ be dualizable. The *continuous K-theory* of $\mathcal{C}$ is the space
$$\mathrm{K}_{\mathrm{cont}}(\mathcal{C}) = \Omega\mathrm{K}(\mathrm{Calk}(\mathcal{C})^\omega).$$

Lemma 5. *If $\mathcal{C}$ is compactly generated, then $\mathrm{K}_{\mathrm{cont}}(\mathcal{C}) = \mathrm{K}(\mathcal{C}^\omega)$.*

Proof. In this case, the localization sequence of Proposition 3 is Ind of the sequence
$$\mathcal{C}^\omega \hookrightarrow \mathcal{C} \to \mathrm{Calk}(\mathcal{C})^\omega.$$
Since $\mathrm{K}(\mathcal{C}) = 0$, the result follows from the localization theorem in K-theory. □

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More generally, we can define a continuous version of any localizing invariant. In the case of K-theory, we made use of the fact that K-theory is defined on the large category $\mathrm{Calk}(\mathcal{C})^\omega$, but this is not the case for an abstract localizing invariant. Thus, we will need to use a presentable version of the Calkin category, which depends on the choice of a large enough regular cardinal.

Definition 6. Let $\mathcal{C}$ be a presentable stable category and κ a regular cardinal. The *κ-Calkin category* $\mathrm{Calk}_\kappa(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$ is the kernel of the colimit functor $\mathrm{colim}\colon \mathrm{Ind}(\mathcal{C}^\kappa) \to \mathcal{C}$.

If $\mathcal{C} \in \mathcal{P}\mathrm{r}_{\mathrm{St}}$ is dualizable, there exists a regular cardinal κ such that the fully faithful functor $\hat{y}\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ lands in $\mathrm{Ind}(\mathcal{C}^\kappa)$. We shall then say that $\mathcal{C}$ is *κ-dualizable*, and we let $\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\kappa\text{-dual}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}}$ be the full subcategory spanned by the κ-dualizable categories. We have

$$\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}} = \bigcup_\kappa \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\kappa\text{-dual}} \quad \text{and} \quad \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\kappa\text{-dual}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\kappa\text{-cg}}.$$

For example, a stable presentable category is ω-dualizable if and only if it is compactly generated. Given $\mathcal{C} \in \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\kappa\text{-dual}}$, we have a localization sequence

$$\mathcal{C} \underset{\mathrm{colim}}{\overset{\hat{y}}{\rightleftarrows}} \mathrm{Ind}(\mathcal{C}^\kappa) \overset{\Phi_\kappa}{\rightleftarrows} \mathrm{Calk}_\kappa(\mathcal{C}) \tag{7}$$

in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$, where Φ_κ is the restriction of Φ, and this diagram is functorial in $\mathcal{C}$. Moreover, it is easy to check that $\mathrm{Calk}_\kappa\colon \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\kappa\text{-dual}} \to \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{cg}}$ preserves localization sequences.

Let $\mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}}$ be the category of small stable idempotent complete categories and exact functors. Recall that Ind-completion induces an isomorphism $\mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}} \simeq \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{cg}}$. If $\mathcal{T}$ is a category, we shall say that a functor $F\colon \mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}} \to \mathcal{T}$ is a *localizing invariant* if it preserves final objects and sends localization sequences to fiber sequences.

Lemma 8. *Let $\mathcal{T}$ be a category and $F\colon \mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}} \to \mathcal{T}$ a localizing invariant. Let $\mathcal{C} \in \mathcal{P}\mathrm{r}_{\mathrm{St}}$ be dualizable and let $\kappa \leq \lambda$ be uncountable regular cardinals such that $\hat{y}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$. Then the inclusion $\mathrm{Calk}_\kappa(\mathcal{C}) \subset \mathrm{Calk}_\lambda(\mathcal{C})$ induces an isomorphism*

$$F(\mathrm{Calk}_\kappa(\mathcal{C})^\omega) \simeq F(\mathrm{Calk}_\lambda(\mathcal{C})^\omega).$$

Proof. Applying Calk_κ to the localization sequence (7) for λ gives a localization sequence

$$\mathrm{Calk}_\kappa(\mathcal{C}) \to \mathrm{Calk}_\kappa(\mathrm{Ind}(\mathcal{C}^\lambda)) \to \mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C})).$$

On the other hand, applying (7) to $\mathrm{Calk}_\lambda(\mathcal{C})$ gives a localization sequence

$$\mathrm{Calk}_\lambda(\mathcal{C}) \to \mathrm{Ind}(\mathrm{Calk}_\lambda(\mathcal{C})^\kappa) \to \mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C})).$$

Hence, we obtain isomorphisms

$$F(\mathrm{Calk}_\kappa(\mathcal{C})^\omega) \simeq \Omega F(\mathrm{Calk}_\kappa(\mathrm{Calk}_\lambda(\mathcal{C}))^\omega) \simeq F(\mathrm{Calk}_\lambda(\mathcal{C})^\omega),$$

natural in λ. This proves the claim. ☐

Given Lemma 8, the following definition makes sense:

Definition 9. Let $\mathcal{T}$ be a category and $F\colon \mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}} \to \mathcal{T}$ a localizing invariant. The *continuous extension* of F is the functor $F_{\mathrm{cont}}\colon \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}} \to \mathcal{T}$ defined by

$$F_{\mathrm{cont}}(\mathcal{C}) = \Omega F(\mathrm{Calk}_\kappa(\mathcal{C})^\omega),$$

where κ is any uncountable regular cardinal such that $\hat{y}(\mathcal{C}) \subset \mathrm{Ind}(\mathcal{C}^\kappa)$.

More formally, we can consider $(\mathrm{Calk}_\kappa)_\kappa$ as a functor

$$\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}} \to \widehat{\mathrm{Ind}}(\mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{cg}}) \simeq \widehat{\mathrm{Ind}}(\mathcal{C}\mathrm{at}_{\mathrm{St}}^{\mathrm{idem}})$$

and compose it with the $\widehat{\mathrm{Ind}}$-extension of ΩF, the result landing in the subcategory $\mathcal{T} \subset \widehat{\mathrm{Ind}}(\mathcal{T})$ by Lemma 8. In the case of K-theory, Definition 9 agrees with Definition 4 since $\mathrm{Calk}(\mathcal{C})^\omega = \mathrm{colim}_\kappa \mathrm{Calk}_\kappa(\mathcal{C})^\omega$ and K-theory commutes with filtered colimits.

We shall say that a functor $F\colon \mathcal{P}\mathrm{r}_{\mathrm{St}}^{\mathrm{dual}} \to \mathcal{T}$ is a *localizing invariant* if it preserves final objects and sends localization sequences to fiber sequences.

Theorem 10 (Efimov)**.** *Let $\mathcal{T}$ be a category. The functor*

$$\mathrm{Fun}(\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}, \mathcal{T}) \to \mathrm{Fun}(\mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}}, \mathcal{T}), \quad F \mapsto F \circ \mathrm{Ind},$$

restricts to an isomorphism between the full subcategories of localizing invariants, with inverse $F \mapsto F_{\mathrm{cont}}$. In particular, if $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}$, then $F_{\mathrm{cont}}(\mathcal{C}) = F(\mathcal{C}^{\omega})$.

Proof. Let $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ be a localization sequence in $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$. Then for large enough κ we have an induced localization sequence

$$\mathrm{Calk}_{\kappa}(\mathcal{A}) \to \mathrm{Calk}_{\kappa}(\mathcal{B}) \to \mathrm{Calk}_{\kappa}(\mathcal{C}).$$

It follows that F_{cont} is a localizing invariant. The proof of Lemma 5 shows that $F_{\mathrm{cont}} \circ \mathrm{Ind} \simeq F$, functorially in F. To $\mathcal{C} \in \mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$ we can associate the filtered diagram of localization sequences

$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C}^{\kappa}) \to \mathrm{Calk}_{\kappa}(\mathcal{C}), \quad \kappa \gg 0,$$

which immediately gives a functorial isomorphism $F \simeq (F \circ \mathrm{Ind})_{\mathrm{cont}}$. □

Example 11. An example of a localizing invariant $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \to \mathrm{Sp}$ is the functor sending a dualizable category to its Euler characteristic. In fact, this functor is the continuous extension of $\mathrm{THH}\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathrm{Sp}$, since $\mathrm{THH}(\mathcal{C})$ is the Euler characteristic of $\mathrm{Ind}(\mathcal{C})$ in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$.

Example 12. Theorem 10 implies the following:

- $\mathrm{THH}_{\mathrm{cont}}\colon \mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \to \mathrm{Sp}$ factors through the stable category CycSp of cyclotomic spectra;
- $\mathrm{TP}_{\mathrm{cont}}(\mathcal{C}) \simeq \mathrm{THH}_{\mathrm{cont}}(\mathcal{C})^{t\mathbb{T}}$ and $\mathrm{TC}_{\mathrm{cont}}(\mathcal{C})$ is the mapping spectrum $\mathrm{Map}_{\mathrm{CycSp}}(\mathbf{1}, \mathrm{THH}_{\mathrm{cont}}(\mathcal{C}))$;
- $\Omega^{\infty}(\mathbb{K}_{\mathrm{cont}}) \simeq \mathrm{K}_{\mathrm{cont}}$, where $\mathbb{K}$ is nonconnective K-theory;
- the cyclotomic trace $\mathbb{K} \to \mathrm{TC}$ extends uniquely to a transformation $\mathbb{K}_{\mathrm{cont}} \to \mathrm{TC}_{\mathrm{cont}}$.

One immediately recovers the following excision theorem of Tamme:

Corollary 13 (Tamme)**.** *Let*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{f} & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{g} & \mathcal{D} \end{array}$$

be a commutative square in $\mathcal{P}\mathrm{r}^{\mathrm{cg}}_{\mathrm{St}}$, cartesian in $\mathcal{C}\mathrm{at}$, such that the right adjoint of g is fully faithful. For any stable category $\mathcal{T}$ and any localizing invariant $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$, the induced square

$$\begin{array}{ccc} F(\mathcal{A}^{\omega}) & \longrightarrow & F(\mathcal{B}^{\omega}) \\ \downarrow & & \downarrow \\ F(\mathcal{C}^{\omega}) & \longrightarrow & F(\mathcal{D}^{\omega}) \end{array}$$

is cartesian.

Proof. Note that the right adjoint of f is also fully faithful. The kernels of f and g are dualizable and isomorphic. Since $\mathcal{T}$ is stable, F_{cont} takes the given square to a cartesian square. □

For the next lemma, note that each subcategory $\mathcal{P}\mathrm{r}^{\kappa\text{-dual}}_{\mathrm{St}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}$ is closed under small colimits, hence $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}} \subset \mathcal{P}\mathrm{r}_{\mathrm{St}}$ is closed under small colimits.

Lemma 14. *Let $\mathcal{T}$ be a category, $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$ a localizing invariant, and $\mathcal{K}$ a small category. Then F preserves $\mathcal{K}$-indexed colimits if and only if F_{cont} preserves $\mathcal{K}$-indexed colimits.*

Proof. This follows easily from (7) and the fact that $\mathrm{Ind}\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{P}\mathrm{r}_{\mathrm{St}}$ preserves colimits. □

Theorem 15. *Let X be a locally compact Hausdorff topological space and $\mathcal{C}$ a dualizable stable presentable category.*

(1) (Lurie) $\mathrm{Shv}(X, \mathcal{C})$ *is dualizable in* $\mathcal{P}\mathrm{r}_{\mathrm{St}}$.

(2) (Efimov) *Suppose that* $\mathrm{Shv}(X)$ *is hypercomplete (for example, X is a topological manifold). Let* $\mathcal{T}$ *be a stable compactly generated category and* $F\colon \mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}} \to \mathcal{T}$ *a localizing invariant that preserves filtered colimits. Then*

$$F_{\mathrm{cont}}(\mathrm{Shv}(X,\mathcal{C})) \simeq \Gamma_{\mathrm{c}}(X, F_{\mathrm{cont}}(\mathcal{C})).$$

In particular,

$$F_{\mathrm{cont}}(\mathrm{Shv}(\mathbb{R}^n,\mathcal{C})) \simeq \Omega^n F_{\mathrm{cont}}(\mathcal{C}).$$

Proof. Let $\mathcal{K}(X)$ denote the poset of compact subsets of X. We write $K \Subset L$ if $K \subset U \subset L$ for some open subset $U \subset X$. Since filtered colimits in $\mathcal{T}$ are left exact, there is a fully faithful embedding

$$e\colon \mathrm{Shv}(X,\mathcal{T}) \hookrightarrow \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{T}), \quad e(\mathcal{F})(K) = \operatorname*{colim}_{K\subset U} \mathcal{F}(U),$$

identifying $\mathrm{Shv}(X,\mathcal{T})$ with the subcategory of functors $\mathcal{G}\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathcal{T}$ such that

(a) $\mathcal{G}(\emptyset) = *$;

(b) for every $K, L \in \mathcal{K}(X)$, the following square is cartesian:

$$\begin{array}{ccc} \mathcal{G}(K\cup L) & \longrightarrow & \mathcal{G}(K) \\ \downarrow & & \downarrow \\ \mathcal{G}(L) & \longrightarrow & \mathcal{G}(K\cap L); \end{array}$$

(c) for every $K \in \mathcal{K}(X)$, the canonical map $\mathcal{G}(K) \to \mathrm{colim}_{K\Subset L}\, \mathcal{G}(L)$ is an isomorphism.

Since $\mathcal{T}$ is stable, e preserves colimits. Let $\mathcal{E} \subset \mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{T})$ be the full subcategory of functors satisfying conditions (a) and (b). Define the endofunctor $\mathcal{G} \mapsto \mathcal{G}^\dagger$ of $\mathrm{Fun}(\mathcal{K}(X)^{\mathrm{op}},\mathcal{T})$ by the formula

$$\mathcal{G}^\dagger(K) = \operatorname*{colim}_{K\Subset L} \mathcal{G}(L).$$

One can easily show that the functor $\mathcal{G} \mapsto \mathcal{G}^\dagger$ sends $\mathcal{E}$ to itself. It follows that the inclusion $e(\mathrm{Shv}(X,\mathcal{T})) \subset \mathcal{E}$ has a right adjoint given by $\mathcal{G} \mapsto \mathcal{G}^\dagger$, which exhibits $\mathrm{Shv}(X,\mathcal{T})$ as a retract of $\mathcal{E}$ in $\mathcal{P}\mathrm{r}_{\mathrm{St}}$. Since $\mathcal{E}$ is clearly compactly generated, this shows that $\mathrm{Shv}(X,\mathcal{T})$ is dualizable. In particular, $\mathrm{Shv}(X,\mathrm{Sp})$ and hence also $\mathrm{Shv}(X,\mathcal{C}) = \mathrm{Shv}(X,\mathrm{Sp}) \otimes \mathcal{C}$ are dualizable.

Consider the functor $\tilde{\mathcal{C}}\colon \mathcal{K}(X)^{\mathrm{op}} \to \mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$ defined by

$$\tilde{\mathcal{C}}(K) = \mathrm{Shv}(K,\mathcal{C}).$$

Since F_{cont} is a stable localizing invariant that preserves filtered colimits (by Lemma 14), $F_{\mathrm{cont}} \circ \tilde{\mathcal{C}}$ satisfies conditions (a)–(c). Hence, $F_{\mathrm{cont}} \circ \tilde{\mathcal{C}} = e(\mathcal{F})$ for some $\mathcal{F} \in \mathrm{Shv}(X,\mathcal{T})$. Let $\underline{\mathcal{C}}$ be the constant presheaf on X with value $\mathcal{C}$. Then the obvious map

$$F_{\mathrm{cont}} \circ \underline{\mathcal{C}} \to \mathcal{F}$$

induces isomorphisms on stalks. Since $\mathrm{Shv}(X)$ is hypercomplete and $\mathcal{T}$ is compactly generated, it exhibits $\mathcal{F}$ as the sheafification of $F_{\mathrm{cont}} \circ \underline{\mathcal{C}}$. It remains to show that

$$\Gamma_{\mathrm{c}}(X,\mathcal{F}) \simeq F_{\mathrm{cont}}(\mathrm{Shv}(X,\mathcal{C})). \tag{16}$$

This is obvious if X is compact. In general, choose a compactification $j\colon X \hookrightarrow \bar{X}$ with closed complement $i\colon X_\infty \hookrightarrow \bar{X}$. Let $\bar{\mathcal{F}}$ be the sheaf on $\bar{X}$ such that $e(\bar{\mathcal{F}}) = F_{\mathrm{cont}} \circ \tilde{\mathcal{C}}$. Then $j^*(\bar{\mathcal{F}}) \simeq \mathcal{F}$ and hence there is a fiber sequence

$$\Gamma_{\mathrm{c}}(X,\mathcal{F}) \to \Gamma(\bar{X},\bar{\mathcal{F}}) \to \Gamma(X_\infty, i^*\bar{\mathcal{F}}).$$

On the other hand, the sequence

$$\mathrm{Shv}(X,\mathcal{C}) \xrightarrow{j_!} \mathrm{Shv}(\bar{X},\mathcal{C}) \xrightarrow{i^*} \mathrm{Shv}(X_\infty,\mathcal{C})$$

is a localization sequence in $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$. Applying F_{cont} proves (16). ∎

Remark 17. Let $\mathcal{E}$ be a small stable rigid symmetric monoidal category, for example Perf_k for some E_∞-ring spectrum k. Then Theorem 10 remains true if we replace $\mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}}$ by $\mathrm{Mod}_{\mathcal{E}}(\mathcal{C}\mathrm{at}^{\mathrm{idem}}_{\mathrm{St}})$ and $\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}}$ by $\mathrm{Mod}_{\mathrm{Ind}(\mathcal{E})}(\mathcal{P}\mathrm{r}^{\mathrm{dual}}_{\mathrm{St}})$. The rigidity of $\mathcal{E}$ ensures that $\hat{y}$ and Φ are $\mathcal{E}$-linear functors. A similar generalization of Theorem 15 holds.