# GERSTEN EXACTNESS FOR MOTIVIC SPECTRA

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The goal of this note is to record the fact that Gabber's proof of the exactness of the Gersten complex [Gab94, CTHK97] applies to motivic spectra without $\mathbb{A}^1$-invariance [AI23, AHI25]. Gersten injectivity in this context first appeared in [EM26, §6] and was generalized to base schemes in [BK26, §3].

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## 1. Deflatability

Let $S$ be a derived scheme.

**Definition 1.1.** Let $\mathcal{S} \subset \mathrm{Sm}_S$ be a full subcategory. A presheaf $F\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{C}$ is called *deflatable on* $\mathcal{S}$ if the following triangle commutes in $\mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathcal{C})$:

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{P}^1, F) & \xrightarrow{j^*} & \mathrm{Hom}(\mathbb{A}^1, F). \\ & {\scriptstyle 0^*}\searrow \quad \nearrow{\scriptstyle \pi^*} & \\ & F & \end{array}$$

Note that this implies $0^* \simeq \infty^*\colon \mathrm{Hom}(\mathbb{P}^1, F) \to F$.

**Example 1.2** (Motivic spectra)**.** Let $E \in \mathrm{MS}(S)$ be a motivic spectrum and let $F = \Omega^\infty_{\mathbb{P}^1} E\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Sp}$. By [AHI25, Corollary 4.11], $F$ is deflatable on $\mathrm{Sm}_S$.

**Lemma 1.3.** *Let $F\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathcal{C}$ be deflatable on $\mathcal{S}$ and suppose $\mathbb{A}^1_X \in \mathcal{S}$ for all $X \in \mathcal{S}$. Then, for every $n \geq 0$, the following triangle commutes in* $\mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathcal{C})$*:*

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{P}^n, F) & \xrightarrow{j^*} & \mathrm{Hom}(\mathbb{A}^n, F). \\ & {\scriptstyle 0^*}\searrow \quad \nearrow{\scriptstyle \pi^*} & \\ & F & \end{array}$$

*Proof.* We proceed by induction on $n$. The statement is trivial when $n = 0$, so let $n \geq 1$. Choose a section $x\colon \mathrm{Spec}(\mathbb{Z}) \to \mathbb{P}^{n-1} \subset \mathbb{P}^n$ in the hyperplane at infinity. The blowup of $\mathbb{P}^n$ at $x$ can be identified with $\mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O}(-1) \oplus \mathcal{O})$ in such a way that the exceptional divisor is the section at infinity and $\mathbb{A}^n$ is the open subscheme $\mathbb{V}_{\mathbb{A}^{n-1}}(\mathcal{O}(-1))$. By deflatability on $\mathcal{S}$, the following diagram commutes naturally in $X \in \mathcal{S}$:

$$\begin{array}{ccccccc} & & F(\mathbb{P}_{\mathbb{P}^{n-1}_X}(\mathcal{O}(-1)\oplus\mathcal{O})) & \to & F(\mathbb{P}_{\mathbb{A}^{n-1}_X}(\mathcal{O}(-1)\oplus\mathcal{O})) & \xrightarrow{j^*} & F(\mathbb{V}_{\mathbb{A}^{n-1}_X}(\mathcal{O}(-1))) \\ & \nearrow & & \searrow{\scriptstyle 0^*} & & \searrow{\scriptstyle 0^*} & \nearrow{\scriptstyle \pi^*} & & \searrow{\scriptstyle \sim} \\ F(\mathbb{P}^n_X) & & \xrightarrow{e^*} & & F(\mathbb{P}^{n-1}_X) & \xrightarrow{j^*} & F(\mathbb{A}^{n-1}_X) & \xrightarrow{\rho^*} & F(\mathbb{A}^n_X). \end{array}$$

Here, $e\colon \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$ is the hyperplane through 0 normal to $x$ and $\rho\colon \mathbb{A}^n \to \mathbb{A}^{n-1}$ is the corresponding projection. We conclude by the induction hypothesis. □

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**Proposition 1.4.** *Let $F\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Set}$ be a Zariski sheaf of sets. The following are equivalent:*

(i) *$F$ is deflatable.*
(ii) *$F$ is deflatable on affines.*
(iii) *For every $X \in \mathrm{Sm}_S$, $F(X) \to F(\mathbb{P}^1 \times X)$ is an isomorphism.*
(iv) *For every $X \in \mathrm{Sm}_S$ and $\mathcal{E} \in \mathrm{Vect}(X)$ of positive rank, $F(X) \to F(\mathbb{P}_X(\mathcal{E}))$ is an isomorphism.*

*Proof.* It is clear that (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) $\Rightarrow$ (ii) for any presheaf $F$. We prove (iv) assuming (ii). The statement is Zariski-local on $X$, so we can assume $X$ affine and $\mathcal{E} = \mathcal{O}^n$. Let $\mathcal{E}_i$ be the summand of $\mathcal{E}$ obtained by removing the $i$th copy of $\mathcal{O}$, so that $\{\mathbb{V}(\mathcal{E}_1), \dots, \mathbb{V}(\mathcal{E}_n)\}$ is an open covering of $\mathbb{P}(\mathcal{E})$. By Zariski descent, $F(\mathbb{P}(\mathcal{E}))$ consists of $n$-tuples

$$(f_1, \dots, f_n) \in F(\mathbb{V}(\mathcal{E}_1)) \times \cdots \times F(\mathbb{V}(\mathcal{E}_n)),$$

such that $f_i = f_j$ over the intersection $\mathbb{V}(\mathcal{E}_i) \cap \mathbb{V}(\mathcal{E}_j)$. By Lemma 1.3 applied with the subcategory of affine schemes $\mathcal{S} \subset \mathrm{Sm}_S$, we have $f_i = \pi_i^*(u_i)$ where $\pi_i\colon \mathbb{V}(\mathcal{E}_i) \to X$ is the projection. Thus, $F(\mathbb{P}(\mathcal{E}))$ consists of $n$-tuples $(u_1, \dots, u_n) \in F(X)^n$ such that $\pi_i^*(u_i) = \pi_j^*(u_j)$ over $\mathbb{V}(\mathcal{E}_i) \cap \mathbb{V}(\mathcal{E}_j)$, which is equivalent to $u_i = u_j$ since there exists a section $X \to \mathbb{V}(\mathcal{E}_i) \cap \mathbb{V}(\mathcal{E}_j)$. □

**Corollary 1.5.** *Let $F\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Set}$ be a deflatable Zariski sheaf of sets. Then:*

(i) *For every $X \in \mathrm{Sm}_S$ and $\mathcal{E} \in \mathrm{Vect}(X)$, the following triangle commutes:*

$$\begin{array}{ccc} F(\mathbb{P}_X(\mathcal{E} \oplus \mathcal{O})) & \xrightarrow{j^*} & F(\mathbb{V}_X(\mathcal{E})). \\ & {\scriptstyle 0^*} \searrow \quad \nearrow {\scriptstyle \pi^*} & \\ & F(X) & \end{array}$$

(ii) *$F$ satisfies elementary blowup excision.*

*Proof.* By Proposition 1.4, $0^*$ is an isomorphism with inverse $p^*$, where $p \circ j = \pi$. Moreover, $F$ sends the blowup square of the zero section $X \hookrightarrow \mathbb{P}_X(\mathcal{E} \oplus \mathcal{O})$ to a square of isomorphisms. □

## 2. The coniveau filtration

Let $X$ be a qcqs topological space. For $n \geq -1$, define the pro-space $X^{\leq n} \in \mathrm{Pro}(\mathrm{Open}^{\mathrm{qc}}(X))$ to be the limit of the quasi-compact open subsets $X - Z$ where $Z$ has codimension $\geq n+1$ (i.e., all irreducible components of $Z$ have codimension $\geq n+1$). This defines a filtration

$$\varnothing = X^{\leq -1} \subset X^{\leq 0} \subset X^{\leq 1} \subset \cdots \subset X.$$

Note that if $X$ has Krull dimension $\leq n$, then $X^{\leq n} = X$.

Let now $F$ be a presheaf of pointed types on $\mathrm{Open}^{\mathrm{qc}}(X)$. We also denote by $F$ the left Kan extension of $F$ to $\mathrm{Pro}(\mathrm{Open}^{\mathrm{qc}}(X))$. For $n \geq -1$, we define the presheaf $F^{\leq n}$ on $\mathrm{Open}^{\mathrm{qc}}(X)$ by $F^{\leq n}(U) = F(U^{\leq n})$, that is,

$$F^{\leq n}(U) = \operatorname*{colim}_Z F(U - Z),$$

where $Z$ runs over all constructible closed subsets of $U$ of codimension $\geq n+1$. This defines a filtration

$$F \to \cdots \to F^{\leq 1} \to F^{\leq 0} \to F^{\leq -1},$$

called the *coniveau filtration* of $F$, with associated graded

$$\mathrm{gr}^n F = \mathrm{fib}(F^{\leq n} \to F^{\leq n-1}).$$

For $n \geq 0$, let $F^{\geq n} = \mathrm{fib}(F \to F^{\leq n-1})$. We then have fiber sequences

$$F^{\geq n+1} \to F^{\geq n} \to \mathrm{gr}^n F. \tag{2.1}$$

The coniveau filtration induces for each $n \geq 0$ a chain complex of pointed presheaves

$$* \to \Omega^n F^{\geq 0} \to \Omega^n \mathrm{gr}^0 F \to \Omega^{n-1} \mathrm{gr}^1 F \to \cdots \to \mathrm{gr}^n F,$$

called the *Gersten complex* for $F$ in degree $n$. Note that $F^{\leq -1} = *$ and $F^{\geq 0} = F$ if $F(\varnothing) = *$.

**Remark 2.2.** Because of the fiber sequence (2.1), the following three statements are equivalent:

- The map $\pi_0 F^{\geq n+1}(X) \to \pi_0 F^{\geq n}(X)$ is zero.
- The map $\pi_0 F^{\geq n}(X) \to \pi_0 \mathrm{gr}^n F(X)$ has trivial kernel.
- The map $\pi_1 \mathrm{gr}^n F(X) \to \pi_0 F^{\geq n+1}(X)$ has trivial cokernel (i.e., the action is transitive).

To describe the associated graded of the coniveau filtration, we need to introduce a finiteness condition on $X$ and some notation:

**Definition 2.3.** Let $X$ be a qcqs space and let $n \geq 0$. We say that $X$ is *finite in codimension $n$* if every constructible subset of $X$ has finitely many irreducible components of codimension $\leq n$.

Note that this holds for all $n$ if $X$ is noetherian, and that $X$ is finite in codimension 0 if and only if it has finitely many irreducible components.

**Notation 2.4.** Let $X$ be a qcqs sober space.

- For $n \geq 0$, let $X^{(n)}$ denote the set of points of codimension $n$ in $X$.
- For $x \in X$, denote by $X_x \in \mathrm{Pro}(\mathrm{Open}^{\mathrm{qc}}(X))$ the localization of $X$ at $x$, i.e., the limit of all quasi-compact open subsets of $X$ that contain $x$.
- For a pointed presheaf $F$ on $\mathrm{Open}^{\mathrm{qc}}(X)$ and a constructible closed subset $Z \subset X$, denote by $F_Z$ the presheaf of sections of $F$ with support in $Z$, defined by
$$F_Z(U) = \mathrm{fib}(F(U) \to F(U - Z)).$$
- For a family of pointed objects $(A_i)_{i \in I}$, define
$$\bigoplus_{i \in I} A_i = \operatorname*{colim}_{J \subset I} \prod_{i \in J} A_i,$$
where $J$ runs over all finite subsets of $I$.

**Proposition 2.5** (Associated graded of the coniveau filtration)**.** *Let $X$ be a spectral space, $F$ a sheaf of pointed types on $X$, and $n \geq 0$. If $X$ is finite in codimension $n$, then:*

(i) *$F^{\leq n}$ is a sheaf on $X$.*

(ii) *$\mathrm{gr}^n F$ is given on quasi-compact open subsets by*
$$U \mapsto \bigoplus_{x \in U^{(n)}} F_{\bar{x}}(U_x).$$

*Proof.* Since $X$ is spectral, every closed subset of $X$ is a cofiltered intersection of constructible closed subsets, which by assumption only have finitely many points of codimension $\leq n$. This implies that, for every $m \leq n+1$, every closed subset of codimension $\geq m$ is a cofiltered intersection of constructible closed subsets of codimension $\geq m$.

(i) Let $U \subset X$ be a quasi-compact open subset and $Z \subset U$ a constructible closed subset of codimension $\geq n+1$. Applying the preliminary observation with $m = n+1$ to the closure of $Z$, we find a constructible closed subset $Z' \subset X$ of codimension $\geq n+1$ such that $Z = Z' \cap U$. Hence, we have $U^{\leq n} = U \cap X^{\leq n}$, which immediately implies that $F^{\leq n}$ is a sheaf on $\mathrm{Open}^{\mathrm{qc}}(X)$.

(ii) We may assume $U = X$. Note that $F_{\bar{x}}(X_x)$ is not a priori well-defined according to Notation 2.4, since $\bar{x}$ need not be constructible in $X$. However, the proof will show that $X_x$ is a limit of quasi-compact open subsets in which $\bar{x}$ is constructible.

By definition,
$$\mathrm{gr}^n F(X) = \operatorname*{colim}_{W \subset Z} F_Z(X - W),$$
where $Z$ runs over constructible closed subsets of codimension $\geq n$ and $W$ over constructible closed subsets of codimension $\geq n+1$. Fix such a $Z$ and let $z_1, \ldots, z_r$ be its points of codimension $n$. As $X$ is spectral, we can write $Z = Z_1 \cup \cdots \cup Z_r$ where $Z_i$ is closed constructible and does not contain $z_j$ for $j \neq i$. Let $Y_i = \bigcup_{j \neq i} Z_j$. The set of $W$'s containing $\bigcap_{i=1}^r Y_i$ is then cofinal, since the latter has codimension $\geq n+1$. For such $W$, we have $\bigcap_{i=1}^r (W \cup Y_i) = W$, and since $F$ is a sheaf, we obtain
$$F_Z(X - W) = \prod_{i=1}^r F_{Z_i}(X - (W \cup Y_i)).$$
Taking the colimit over $W$ on the right-hand side, we claim that
$$\operatorname*{colim}_W F_{Z_i}(X - (W \cup Y_i)) = F_{\bar{z}_i}(X_{z_i}).$$
Indeed, among the quasi-compact open subsets $U$ containing $z_i$, the ones disjoint from $Y_i$ are cofinal. For such $U$, we have $Z_i \cap U = \bar{z}_i \cap U$, and if we let $W = Z_i - U$, then the inclusion $U \subset X - (W \cup Y_i)$ induces an isomorphism on $F_{Z_i}$ since $F$ is a sheaf. Altogether, we get
$$\operatorname*{colim}_W F_Z(X - W) = \prod_{i=1}^r F_{\bar{z}_i}(X_{z_i}).$$
Taking the colimit over $Z$ gives the desired formula, since every point of $X^{(n)}$ is a maximal point of such a $Z$ (by the preliminary observation with $m = n$). □

## 3. Exactness of the Gersten complex

Let $F\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Typ}_*$ be a presheaf of pointed types. We consider the following deflatability condition on $F$, which is a variant of [CTHK97, §5.1 Axiom COH2]:

($*$) For every $d \geq 0$, affine open $X \subset \mathbb{A}^d_S$, and closed subset $Z \subset X$, the following triangle commutes:

$$\begin{array}{ccc} \pi_0 F_{\mathbb{P}^1_Z}(\mathbb{P}^1_X) & \xrightarrow{j^*} & \pi_0 F_{\mathbb{A}^1_Z}(\mathbb{A}^1_X). \\ & {\scriptstyle 0^*}\searrow \quad \nearrow{\scriptstyle \pi^*} & \\ & \pi_0 F_Z(X) & \end{array}$$

Note that $F$ satisfies ($*$) if it is deflatable on open subschemes of affine spaces.

**Proposition 3.1** (Gabber's effacement theorem)**.** *Let $k$ be a field and let $F\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Typ}_*$ be a Nisnevich sheaf of pointed types. Let $X$ be a smooth affine $k$-scheme, $x_1, \dots, x_r \in X$, and $Z \subset X$ a closed subset of codimension $\geq 1$. If $k$ is finite, assume $r = 1$. Then there exists an open subset $U \subset X$ containing $x_1, \dots, x_r$ and a closed subset $Y \subset U$ containing $Z \cap U$ in codimension 1 such that:*

(i) *If $F$ is deflatable on open subschemes of affine spaces, then the map*
$$F_Z(U) \to F_Y(U)$$
*is nullhomotopic.*

(ii) *If $F$ satisfies ($*$), then the map*
$$\pi_0 F_Z(U) \to \pi_0 F_Y(U)$$
*is zero.*

(iii) *If $\pi_0 F$ is deflatable on affine open subschemes of affine spaces, then the map*
$$\pi_0 F_Z(U) \to \pi_0 F(U)$$
*is zero.*

*Proof.* By Gabber's preparation lemma ([CTHK97, Theorem 3.1.1] if $k$ is infinite, [HK20, Theorem 1.1] if $k$ is finite), there exists an open neighborhhod $U$ of $x_1, \dots, x_r$ and an étale map $\varphi\colon U \to \mathbb{A}^1_V$ over $k$, where $V$ is open in some affine space, such that $T = \varphi(Z \cap U)$ is closed in $\mathbb{P}^1_V$ and $\varphi$ is is an isomorphism over $T$. Shrinking $V$ if necessary, we can assume that it is affine. Let $Y = \varphi^{-1}(\mathbb{A}^1_{\pi(T)})$. In the category of pointed Nisnevich sheaves on $\mathrm{Sm}_k$, we then have a commutative diagram

$$\begin{array}{ccccc} \dfrac{U}{U-Y} & & \longrightarrow & & \dfrac{U}{U-Z} \\ {\scriptstyle\varphi}\downarrow & & & & \wr\downarrow{\scriptstyle\varphi} \\ \dfrac{\mathbb{A}^1_V}{\mathbb{A}^1_{V-\pi(T)}} & \longrightarrow & \dfrac{\mathbb{P}^1_V}{\mathbb{P}^1_{V-\pi(T)}} & \longrightarrow & \dfrac{\mathbb{P}^1_V}{\mathbb{P}^1_V - T}, \end{array}$$

where the right vertical map is an isomorphism. Under the assumption of (i) resp. (ii), the bottom row is zero after applying $\mathrm{Map}(-, F)$ resp. $\pi_0\mathrm{Map}(-, F)$, hence so is the top row. Under the assumption of (iii), $\pi_0 F_T(\mathbb{P}^1_V) \to \pi_0 F(\mathbb{A}^1_V)$ is zero, and hence $\pi_0 F_Z(U) \to \pi_0 F(U)$ is zero. □

We will say that a $k$-scheme $X$ is *openly pro-smooth* if it is the limit of a cofiltered diagram of smooth $k$-schemes with open affine transition maps. Given a presheaf $F$ on $\mathrm{Sm}_k$, we can as usual evaluate $F$ on any qcqs pro-smooth $k$-scheme $X$ (and in particular on $\mathrm{Open}^{\mathrm{qc}}(X)$) by left Kan extension.

**Corollary 3.2.** *Let $k$ be a field, let $F\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Typ}_*$ be a Nisnevich sheaf of pointed types, and let $n \geq 0$. Let $X$ be a pro-smooth semilocal $k$-scheme. If $k$ is finite, assume $X$ local. Then the map*
$$\pi_0 F^{\geq n+1}(X) \to \pi_0 F^{\geq n}(X)$$
*is zero if either of the following conditions holds:*

(i) *$X$ is openly pro-smooth and $F$ satisfies ($*$).*

(ii) *$n = 0$ and $\pi_0 F$ is deflatable on affine open subschemes of affine spaces.*

*Proof.* Write $X$ as a cofiltered limit of smooth affine $k$-schemes $X_\alpha$, and denote by $\mathrm{Z}^{\geq n}(X)$ the poset of constructible closed subsets of $X$ of codimension $\geq n$. If the transition maps are open, then there is an isomorphism
$$\operatorname*{colim}_\alpha \mathrm{Z}^{\geq n}(X_\alpha) \xrightarrow{\sim} \mathrm{Z}^{\geq n}(X),$$

so that $F^{\geq n+1}(X) = \operatorname{colim}_\alpha F^{\geq n+1}(X_\alpha)$. Assertion (i) now follows from Proposition 3.1(ii). To prove Assertion (ii), we may assume $X$ and $X_\alpha$ integral. Let $\eta$ be the generic point of $X$ and let $\eta_\alpha$ be the localization of $X_\alpha$ at the image of $\eta$, so that $\eta = \lim_\alpha \eta_\alpha$. By Proposition 3.1(iii) and Remark 2.2, each map $\pi_0 F(X_\alpha) \to \pi_0 F(\eta_\alpha)$ has trivial kernel, which implies that $\pi_0 F(X) \to \pi_0 F(\eta)$ has trivial kernel, as desired. □

**Theorem 3.3** (Exactness of the Gersten complex)**.** *Let $k$ be a field, let $F\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Typ}_*$ be a Nisnevich sheaf of pointed types, and let $n \geq 0$. Let $X$ be a pro-smooth semilocal $k$-scheme. If $k$ is finite, assume $X$ local. Consider the Gersten complex for $F$ in degree $n$:*

$$* \to \pi_n F(X) \to \pi_n \mathrm{gr}^0 F(X) \to \pi_{n-1}\mathrm{gr}^1 F(X) \to \cdots \to \pi_1 \mathrm{gr}^{n-1} F(X) \to \pi_0 \mathrm{gr}^n F(X).$$

*If $X$ is openly pro-smooth and $F$ is deflatable on open subschemes of affine spaces, then the complex is exact. More precisely:*

(i) (Gersten injectivity) *If $\pi_n F$ is deflatable on affine open subschemes of affine spaces, then the complex is exact at $\pi_n F(X)$.*
(ii) (Purity in codimension one) *If $X$ is openly pro-smooth and $\Omega^{n-1}F$ satisfies $(*)$, then the complex is exact at $\pi_n \mathrm{gr}^0 F(X)$.*
(iii) (Purity in higher codimension) *Let $1 \leq i \leq n-1$. If $X$ is openly pro-smooth and both $\Omega^{n-i-1}F$ and $\Omega^{n-i}F$ satisfy $(*)$, then the complex is exact at $\pi_{n-i}\mathrm{gr}^i F(X)$.*

*Proof.* (i) is exactly Corollary 3.2(ii) applied to $\Omega^n F$, while (ii,iii) follow from Corollary 3.2(i): the exactness at $\pi_{n-i}\mathrm{gr}^i F(X)$ follows from the triviality of the kernel of $\pi_{n-i-1}F^{\geq i+1}(X) \to \pi_{n-i-1}\mathrm{gr}^{i+1}F(X)$ and, if $i \geq 1$, from the surjectivity of $\pi_{n-i+1}\mathrm{gr}^{i-1}F(X) \to \pi_{n-i}F^{\geq i}(X)$:

$$\begin{array}{ccccc} & & \pi_{n-i-1}F^{\geq i+1} & & \\ & \nearrow & & \searrow & \\ \pi_{n-i+1}\mathrm{gr}^{i-1}F & \longrightarrow & \pi_{n-i}\mathrm{gr}^i F & \longrightarrow & \pi_{n-i-1}\mathrm{gr}^{i+1}F. \\ & \searrow & & \nearrow & \\ & & \pi_{n-i}F^{\geq i} & & \end{array}$$

□

**Corollary 3.4.** *Let $k$ be a field, let $F\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Typ}_*$ be a Nisnevich sheaf of pointed types that is deflatable on open subschemes of affine spaces, and let $n \geq 0$. Let $X$ be a pro-smooth semilocal $k$-scheme. If $k$ is finite, assume $X$ local.*

(i) *The map*

$$\pi_n F(X) \to \bigoplus_{x \in X^{(0)}} \pi_n F(x)$$

*has trivial kernel*

(ii) *If $X$ is openly pro-smooth and finite in codimension $n$, then the Gersten complex*

$$* \to \pi_n F(X) \to \bigoplus_{x\in X^{(0)}} \pi_n F(x) \to \bigoplus_{x\in X^{(1)}} \pi_{n-1} F_{\bar{x}}(X_x) \to \cdots \to \bigoplus_{x\in X^{(n)}} \pi_0 F_{\bar{x}}(X_x)$$

*is exact.*

*Proof.* Combine Theorem 3.3 with Proposition 2.5, noting that $X$ is finite in codimension 0. □

**Example 3.5** (Perfect valuation rings)**.** Corollary 3.4(i) applies if $k$ is perfect and $X$ is a perfect valuation ring over $k$, by [EHIK21, Theorem 3.4.5].

## 4. Contraction

**Definition 4.1.** A Nisnevich sheaf of abelian groups $F\colon \mathrm{Sm}_S^{\mathrm{op}} \to \mathrm{Ab}$ is called *strictly deflatable* if, for every $i \geq 0$, the presheaf of abelian groups $\mathrm{H}^i_{\mathrm{Nis}}(-, F)$ is deflatable. We denote by

$$\mathrm{Shv}_{\mathrm{s}\mathbb{P}}(\mathrm{Sm}_S, \mathrm{Ab}) \subset \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S, \mathrm{Ab})$$

the full subcategory of strictly deflatable sheaves.

**Proposition 4.2.** *Let $k$ be a field and let $F\colon \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Sp}$ be a Nisnevich sheaf in the heart of the t-structure such that $\pi_0 F$ is strictly deflatable. Then $\Omega_{\Sigma^{-1}\mathbb{P}^1}F$ also lies in the heart of the t-structure and $\pi_0\Omega_{\Sigma^{-1}\mathbb{P}^1}F$ is strictly deflatable.*

*Proof.* Note that $\pi_*\Omega_{\Sigma^{-1}\mathbb{P}^1}F = \Omega_{\mathbb{P}^1}\pi_{*-1}F$, which is deflatable as $\pi_*F$ is deflatable. Let $X$ be a henselian localization of a smooth $k$-scheme with function field $K$. It remains to show that $\pi_i(\Omega_{\mathbb{P}^1}F)(X)$ vanishes for $i \neq -1$. It vanishes for $i > 0$ as $F$ is 0-truncated. Since $\Omega_{\mathbb{P}^1}F$ is a Nisnevich sheaf such that $\pi_i\Omega_{\mathbb{P}^1}F$ is deflatable, we have

$$\pi_i(\Omega_{\mathbb{P}^1}F)(X) \subset \pi_i(\Omega_{\mathbb{P}^1}F)(\operatorname{Spec} K) \subset \pi_iF(\mathbb{P}^1_K)$$

by Gersten injectivity (Theorem 3.3(i)). This vanishes for $i \leq -2$ since the Nisnevich topos of $\mathbb{P}^1_K$ has homotopy dimension $\leq 1$. Finally, $\pi_0(\Omega_{\mathbb{P}^1}F)(X)$ is the kernel of

$$\infty^*\colon \pi_0F(\mathbb{P}^1_X) \to \pi_0F(X),$$

which is an isomorphism by Proposition 1.4, as $\pi_0F$ is a deflatable Zariski sheaf. □

By Proposition 4.2, there is a unique endofunctor

$$\mathrm{Shv}_{\mathrm{s}\mathbb{P}}(\mathrm{Sm}_k, \mathrm{Ab}) \to \mathrm{Shv}_{\mathrm{s}\mathbb{P}}(\mathrm{Sm}_k, \mathrm{Ab}), \quad A \mapsto A_{-1},$$

called *contraction*, making the following square commute, where the vertical functors are the inclusion of the heart of the t-structure:

$$\begin{array}{ccc} \mathrm{Shv}_{\mathrm{s}\mathbb{P}}(\mathrm{Sm}_k, \mathrm{Ab}) & \xrightarrow{A \mapsto A_{-1}} & \mathrm{Shv}_{\mathrm{s}\mathbb{P}}(\mathrm{Sm}_k, \mathrm{Ab}) \\ \downarrow & & \downarrow \\ \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp}) & \xrightarrow{\Omega_{\Sigma^{-1}\mathbb{P}^1}} & \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_k, \mathrm{Sp}). \end{array}$$

This endofunctor also extends the usual contraction functor $\Omega_{\mathbb{G}_m}$ on the subcategory $\mathrm{Shv}_{\mathrm{s}\mathbb{A}}(\mathrm{Sm}_S, \mathrm{Ab}) \subset \mathrm{Shv}_{\mathrm{s}\mathbb{P}}(\mathrm{Sm}_S, \mathrm{Ab})$ of strictly $\mathbb{A}^1$-invariant sheaves.

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